

RIESZ DISTRIBUTIONS ASSOCIATED TO DUNKL OPERATORS

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Dunkl operators are a deformation of the usual derivatives by certain non-local operators that involve reflections from a given root system, and remarkably they (for different directions) commute with each other. For the last 25 years or so, they have been studied extensively, on both the analytical and algebraic sides, and old connections and new applications have been found in many disparate fields of mathematics. One of the highlights was a Huygens Principle for the Dunkl wave equation, a phenomenon with a long history in the differential case. The theory of Riesz distributions associated to Dunkl operators is developed here in order to shed new light on the Dunkl wave equation. The techniques are also applied to a higher order hyperbolic equation. These equations are very special for they have to satisfy a Bernstein-Sato type identity with Dunkl operators, and one may attribute the phenomenon of Huygens Principle to the fact that Dunkl operators shift the roots of the Bernstein-Sato polynomial.

BIOGRAPHICAL SKETCH

Yao LIU was born on January 27, 1986, in Beijing, China. He grew up in the district known for the universities and the Summer Palace, and was fortunate to spend his formative years in the High School Affiliated to Peking University. At the age of 15 he moved to the United States with his mother, and finished his high school in Lexington, Massachusetts, having received the best secondary education from both China and the U.S. He studied mathematics and physics at Columbia University in the City of New York, graduating with a Bachelor of Arts in 2008, and was very grateful to have had some exposure to the humanities, particularly the philosophical tradition of the West. He then entered the physics PhD program at Cornell University, but soon found himself more attracted to mathematics. With the help of the faculty, he was able to switch to the mathematics program, and came under the tutelage of Yuri Berest. In the last few years he was joined by his wife Jiajia XU, and their lovely son Guangchen (meaning light and dust) who was born in Ithaca.

He enjoys Classical music and plays the violin, and maintains an interest in the recent advances in the sciences, as well as all things pertaining to ancient or contemporary China.

Dedicated to my grandfather YEN Lung-fei (1919 – 2001)
and to my parents LIU Xu and YAN Lin

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CHAPTER 1

INTRODUCTION

Riesz distributions were introduced by Marcel Riesz as an elegant technical tool to understand the phenomenon known as the Huygens Principle. We begin with a brief history of the Huygens Principle as a motivation for our study.

1.1 The Huygens Principle

One of the most fascinating features of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^{n-1}$$

is that the (unique) fundamental solution $E_+ \in \mathcal{S}'(\mathbb{R}^n)$ supported in the future cone $\bar{C}_+ = \{(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1} : |x| \leq t\}$, is actually supported *on* the lightcone ∂C_+ for $n = 4, 6, 8, \dots$. This is what Huygens implicitly “proposed” (for $n = 4$) as the foundation of his wave theory of light (see historical remarks in [Dui91]), but was not put in precise terms until Hadamard formulated it as the “minor premise” of a syllogism in his lectures on Cauchy’s problem [Ha23], anticipating the language of distributions introduced by Schwartz. This property has since been referred to as Huygens Principle¹, at least in the mathematical literature.

In [Ha23], Hadamard wanted to find other second-order hyperbolic equations, i.e., the wave equation with lower order terms and/or variable coefficients, that exhibit this remarkable *local* property. He proved the necessary condition that n be even, and gave a sufficient condition which, however, is highly ineffective to the point that no non-trivial examples could be found from it. By trivial examples we

¹Though more commonly spelled as Huygens’ Principle, I shall follow the modern trend of reducing ’s in mathematical terminology, in consistence with Dunkl operators and Cherednik algebras.

mean those that could be obtained from the pure wave equation by a nonsingular change of variables or multiplication by a non-zero function. The belief that no non-trivial equations with Huygens Principle exist became known as Hadamard's Conjecture, and for $n = 4$ it was proved independently by Mathisson [Ma39] [Ha45] and Ásgeirsson [Ás56]. It then came as a surprise when Stellmacher [St53] found the first counterexample in $n = 6$, with a rational potential:

$$\mathcal{L} = \partial_t^2 - \Delta + \frac{2}{x_1^2}$$

As more examples were being found, it became clear that these are extremely rare occurrences that exhibit interesting algebraic structures that also arise in other areas of mathematics, most notably integrable systems (see the survey [BV94']). One particularly interesting family of examples come from Calogero-Moser systems with *integral* parameters, of which Stellmacher's example is the simplest case. It takes the form

$$\mathcal{L}_k = \partial_t^2 - \Delta + \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha(k_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}$$

and Berest and Veselov [BV94] proved that the Huygens Principle holds when n is even, $k_\alpha \in \mathbb{Z}_{\geq 0}$ and

$$n \geq 4 + 2 \sum_{\alpha \in \mathcal{R}_+} k_\alpha$$

Here $\mathcal{R} \subset \mathbb{R}^{n-1}$ is the root system for a finite reflection group W (Coxeter group) and k_α are a set of parameters satisfying $k_\alpha = k_{w(\alpha)}$ for all $w \in W$. For more history on Huygens Principle, and a renewed Hadamard Conjecture (still open), see [Be98].

On the global side, Huygens Principle was established on odd-dimensional spheres, as well as other symmetric spaces (both compact and non-compact). We shall only mention the names of Lax, Phillips [LP78] and Helgason [Hel91].

In a different direction, Petrovsky [Pe45] took up the study of higher-order analogues of the wave equation, the so-called (strongly) hyperbolic differential equations (with constant coefficients). He gave a necessary and sufficient condition for the vanishing of the fundamental solution in a domain, called a (strong) *lacuna*, outside the *wavefront*, and provided some non-trivial examples. However, the condition turns the problem of lacunas into a non-trivial problem in the topology of real algebraic varieties, and moreover it does not apply to the most interesting examples found by Gårding [Gå47], for which the fundamental solution is supported on a subvariety *within* the wavefront (of very low dimension in fact). About two decades later, Petrovsky’s theory was clarified and further developed by Atiyah, Bott, and Gårding [ABG70] [ABG73] with the new tools from algebraic geometry and algebraic topology, while the examples of Gårding were put in the larger context of homogeneous cones by Gindikin [Gi64] [VG67] [Gi92] — all within the realm of constant coefficients. It should be noted that Gårding, and then Gindikin, were directly building on M. Riesz’s work [Rie49] on the wave equation by means of analytic continuation, which shall be our focus (more details in the next section).

Another development, inspired in part by (modern) algebraic geometry and homological algebra, was the study of systems of linear differential equations as \mathcal{D} -modules, after Bernstein [Ber71] and the Sato school in Japan. Although highly sophisticated techniques have been developed (by Kashiwara and others) to address, among other things, the well-posedness of hyperbolic differential equations with holomorphic coefficients (see [Sc13]), little could be said about Huygens Principle or lacunas. The same goes with the deep and active field of *nonlinear* hyperbolic PDEs, of which Einstein’s equations of general relativity is perhaps the best-known example.

This dissertation was motivated by a recent paper of Ben Said-Ørsted [BØ05], which studies a Huygens Principle of the so-called Dunkl wave operator, which is a differential-reflection operator also associated with a root system \mathcal{R} and parameters $k_\alpha \geq 0$, $\alpha \in \mathcal{R}$ (see below). They found that the fundamental solution (naturally defined, though with no explicit formula) is supported in ∂C_+ if and only if

$$n + 2 \sum_{\alpha \in \mathcal{R}_+} k_\alpha = 4, 6, 8, \dots$$

Note in particular that n needs *not* be even, in contrast with Hadamard's and Petrovsky's conditions for the differential case; and the k_α 's need not be integers, so long as the combination $n + 2 \sum k_\alpha$ satisfies the condition above.

We introduce Riesz distributions associated to Dunkl operators as a new approach to study Dunkl wave operators. It leads quickly to a new proof of Ben Said-Ørsted that is significantly simpler and more transparent, and it also gives an explicit expression of the fundamental solution. The technique used is a modern rendition of Riesz's method of analytic continuation and is almost completely elementary, circumventing the heavy use of Dunkl analysis (particularly the analogue of Fourier transform and a Paley-Wiener type theorem) and the use of oscillator representation and Howe's dual pair $(O_{1,n-1}, \widetilde{SL}(2, \mathbb{R}))$ which is somewhat behind the scene (see the review to [BØ05] in *Math. Rev.* by Schempp).

1.2 Riesz distributions and the Dunkl analogue

To set the stage, we begin by recalling, in some detail, the original construction of M. Riesz [Rie49] (see also [KV91] [DK10]) generalizing the so-called Riemann-Liouville integrals, which is a theory of fractional differentiation and integration in

one dimension. It subsumes the classical solutions of d'Alembert, Poisson, Kirchhoff, etc. in a single explicit formula. For each $s \in \mathbb{C}$ with $\operatorname{Re} s > \frac{n}{2} - 1$, define a tempered distribution $R^s \in \mathcal{S}'(\mathbb{R}^n)$ ² by the regular function

$$R^s(t, x) := \begin{cases} \frac{1}{\Gamma_n(s)} (t^2 - |x|^2)^{s - \frac{n}{2}} & (t, x) \in C_+ \\ 0 & (t, x) \notin C_+ \end{cases}$$

where

$$\Gamma_n(s) := 2^{2s-1} \pi^{\frac{n-2}{2}} \Gamma(s) \Gamma(s - \frac{n}{2} + 1)$$

Then we have a functional equation $(\partial_t^2 - \Delta)R^s = R^{s-1}$ for $\operatorname{Re} s > \frac{n}{2}$, which in turn is used to analytically continue R^s to all $s \in \mathbb{C}$ as a distribution-valued entire function $s \mapsto R^s$. The crucial point is that $R^0 = \delta$, which makes R^1 exactly the fundamental solution E_+ supported in \bar{C}_+ . Furthermore, this family of distributions form a convolution algebra: $R^s * R^t = R^{s+t}$ for all $s, t \in \mathbb{C}$, and one may interpret them as the complex powers of the wave operator:

$$R^s = “ (\partial_t^2 - \Delta)^{-s} ”$$

with respect to the cone C_+ .

Now, it is clear that $R^{\frac{n}{2}}$ is constant inside (as well as outside) C_+ , so $R^{\frac{n}{2}-1}$, being a derivative of $R^{\frac{n}{2}}$, vanishes in the interior of C_+ ; in fact, it is a “delta function” supported along ∂C_+ (except for $n = 2$). Now, if n is even and ≥ 4 , applying $\partial_t^2 - \Delta$ to $R^{\frac{n}{2}-1}$ enough times will reach $R^1 = E_+$. Since differential operators do not enlarge supports, we have established the Huygens Principle for even $n \geq 4$.

On the other hand, if n is odd or $n = 2$, $R^1 = E_+$ restricted to the interior of

²Note the superscript in R^s is merely an index, not to be taken as an exponent.

C_+ coincides with the function

$$\frac{1}{\Gamma_n(1)}(t^2 - |x|^2)^{1-\frac{n}{2}} \not\equiv 0$$

so the Huygens Principle does not hold.

Note that, even though Riesz distributions could be defined for any hyperbolic differential operator $P(\partial)$ via

$$R^s(x) = \frac{1}{(2\pi)^n} \int \frac{e^{i\langle x, \xi - i\eta \rangle}}{P(i(\xi - i\eta))^s} d\xi$$

with any $\eta \in C_+$ (the hyperbolicity cone), this is antithetical to Riesz' original intent to avoid using the Fourier transform.

We now turn to our main results. Let $\mathcal{R} \subset \mathbb{R}^{n-1}$ be a (non-crystallographic) root system associated with a Coxeter group W , and for each $\alpha \in \mathcal{R}$ there is a parameter $k_\alpha \in \mathbb{C}$ satisfying $k_\alpha = k_{w(\alpha)}$ for all $w \in W$. Let s_α be the reflection across the hyperplane perpendicular to α , i.e.,

$$s_\alpha(x) = x - 2 \frac{(\alpha, x)}{(\alpha, \alpha)} \alpha$$

where (\cdot, \cdot) is the standard inner product in \mathbb{R}^{n-1} . Define the *Dunkl operator* $D_i = D_i(W, k)$ by

$$D_i f(x) = \frac{\partial}{\partial x_i} f(x) + \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha \alpha_i}{(\alpha, x)} (f(x) - f(s_\alpha(x))) \quad i = 1, \dots, n-1$$

where α_i is the i -th coordinate of the root α , and \mathcal{R}_+ (any) choice of positive roots in \mathcal{R} . The crucial property due to Dunkl [Du89] is that $D_i D_j = D_j D_i$ (see §2.2 for a coordinate-free definition and a quick proof of the commutativity). The *Dunkl wave operator* is then defined by

$$\partial_t^2 - \Delta_k := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n-1} D_i^2$$

For fixed k with $\operatorname{Re} k_\alpha > -\frac{1}{2}$, the Riesz distributions $R_k^s \in \mathcal{S}'(\mathbb{R}^n)$ associated with W and k are defined, for $\operatorname{Re} s > \frac{n}{2} + \operatorname{Re} \gamma_k - 1$, $\gamma_k = \sum_{\alpha \in \mathcal{R}_+} k_\alpha$, by the regular function

$$R_k^s(t, x) = \frac{1}{\Gamma_{n,k}(s)} (t^2 - |x|^2)^{s - \frac{n}{2} - \gamma_k} v_k(x) \quad (t, x) \in C_+$$

(and vanishes outside C_+), where

$$\Gamma_{n,k}(s) = c_{n,k} 4^s \Gamma(s) \Gamma(s - \frac{n}{2} - \gamma_k + 1)$$

and

$$v_k(x) = \prod_{\alpha \in \mathcal{R}_+} |(\alpha, x)|^{2k_\alpha}$$

(The constant $c_{n,k}$ does not depend on s , and is chosen to normalize the Riesz distributions so that $R_k^0 = \delta$ upon analytic continuation.)

Theorem 1.2.1. *The family of distributions R_k^s is analytic in s (i.e., for each $\phi \in \mathcal{S}(\mathbb{R}^n)$, the number $\langle R_k^s, \phi \rangle$ is analytic in s) for $\operatorname{Re} s > \frac{n}{2} + \operatorname{Re} \gamma_k - 1$, and can be analytically continued to all $s \in \mathbb{C}$ satisfying*

1. $(\partial_t^2 - \Delta_k) R_k^{s+1} = R_k^s$
2. $R_k^0 = \delta$
3. $\operatorname{supp} R_k^{s_0} = \partial C_+$ for $s_0 = \frac{n}{2} + \gamma_k - 1$ (except when $s_0 = 0$)

As an immediate corollary, we have

Corollary 1.2.2. *The fundamental solution to $(\partial_t^2 - \Delta_k)^m$ supported in \bar{C}_+ is given by R_k^m , and*

$$\operatorname{supp} R_k^m \subseteq \partial C_+ \quad \text{if} \quad \frac{n-2}{2} + \gamma_k \in m + \mathbb{Z}_{\geq 0}$$

(which one may regard as a Huygens Principle for differential reflection operators).

Up to this point, neither the definition of Riesz distributions nor the proof (except for the determination of normalization) rely on “Dunkl analysis”. It’s only with the associated Cauchy problem that we need to introduce Dunkl convolution, i.e., $f *_k g$ for suitable functions or distributions f and g , and for “regular” values k , such that

$$D_i(f *_k g) = (D_i f) *_k g = f *_k (D_i g)$$

and

$$f *_k g = g *_k f$$

Unfortunately the standard definition (as in [BØ05]) is not as explicit as one would like. An alternative approach is given in §2.6 by means of Sato’s hyperfunctions, which may be more intuitive, if still not explicit.

Nonetheless, with Dunkl convolution and the fundamental solution R_k^m , one immediately solves the Cauchy problem

$$\begin{cases} (\partial_t^2 - \Delta_k)^m u = 0 \\ \partial_t^i u|_{t=0} = f_i \quad i = 0, 1, \dots, 2m-1 \end{cases}$$

with $2m$ initial conditions in the standard way. If all but the last initial data vanish, the solution (for $t > 0$) is simply given by

$$u(t, x) = R_k^m(t, x) *_k f_{2m-1}(x)$$

where the Dunkl convolution is taken over the x variables.

The connection (for $m = 1$) with the differential operators \mathcal{L}_k in Berest-Veselov is quite subtle. First of all, it’s best to consider the equation $\mathcal{L}_k u = 0$ over the Weyl chamber $\Omega = \{x \in \mathbb{R}^{n-1} \mid (\alpha, x) > 0 \text{ for all } \alpha \in \Pi\}$ ($\Pi \subset \mathcal{R}_+$ being the set of simple roots), and boundary conditions on the “walls” need to be imposed to

ensure a global solution (for all t). Let's state the result for Neumann boundary conditions (this is essentially in [BØ05]):

Theorem 1.2.3.

$$\begin{cases} \mathcal{L}_k u = 0 & \text{on } \mathbb{R} \times \Omega \\ \partial_t^i u|_{t=0} = f_i & i = 0, 1 \\ \partial_\alpha u|_{\alpha=0} = 0 & \forall \alpha \in \Pi \end{cases}$$

has a unique global solution for given $f_i \in \mathcal{D}'(\Omega)$. Furthermore, if f_i are supported in a compact set $S \subset \Omega$, then the solution u vanishes on

$$\{(t, x) \in \mathbb{R}^n : |t| > \sup_{w \in W, y \in S} |x - w(y)|\}$$

provided that

$$n + 2\gamma_k \in \{4, 6, 8, \dots\}$$

What Berest-Veselov gives is a local solution (more precisely, only for $|t| < \inf_{\alpha \in \Pi, y \in S} (\alpha, y)$), but is stronger in the sense that the solution vanishes on a larger domain:

$$\{(t, x) \in \mathbb{R}^n : |t| > \sup_{y \in S} |x - y|\}$$

provided that n is even, $k_\alpha \in \mathbb{Z}_{\geq 0}$, and

$$n \geq 4 + 2\gamma_k$$

We give another application of the techniques of Riesz distributions associated to Dunkl operators. Let $W = S_n$ act on \mathbb{R}^n by permuting the coordinates, and fix a parameter $k \in \mathbb{C}$ with $\operatorname{Re} k > -\frac{1}{2}$. For each $s \in \mathbb{C}$ with $\operatorname{Re} s > (n-1)\operatorname{Re} k$,

define $R_k^s \in \mathcal{S}'(\mathbb{R}^n)$ by the regular function

$$R_k^s = \frac{1}{c \prod_{j=0}^{n-1} \Gamma(s - jk)} (x_1 \dots x_n)^{s-1-(n-1)k} v_k(x)$$

on the first orthant $\bar{C}_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i\}$, where

$$v_k(x) = \prod_{i < j} |x_i - x_j|^{2k}$$

Let X_j be the (closed) j -dimensional stratum of the cone \bar{C}_+ , i.e.

$$X_j := \bigcup_{I \subset \{1, \dots, n\}, |I|=j} \{x \in \mathbb{R}^n \mid x_i = 0 \text{ for all } i \notin I \text{ and } x_i \geq 0 \text{ for all } i \in I\}$$

then,

Theorem 1.2.4. *The distribution-valued function $s \mapsto R_k^s$ is analytic for $\operatorname{Re} s > (n-1)\operatorname{Re} k$, and it can be analytically continued to all $s \in \mathbb{C}$ satisfying*

1. $D_1 \dots D_n R_k^{s+1} = R_k^s$
2. $R_k^0 = \delta$
3. $\operatorname{supp} R_k^{jk} = X_j$ for $j = 0, \dots, n-1$ (except when $k = 0$)

Corollary 1.2.5. *If $k = \frac{p}{q} \in \mathbb{Q} > 0$ with $(p, q) = 1$ and $1 \leq q \leq n-1$, then the Huygens Principle holds for the operator $D_1 \dots D_n$. More precisely, $\operatorname{supp} R_k^1 = X_q$.*

For the case of $k = 0$, the fundamental solution is simply the characteristic function of C_+ , hence the Huygens Principle does not hold. However, it has a partial Huygens Principle in the sense that if the “last” initial condition $\partial_t^{n-1} u$ on the “Cauchy surface” vanishes, then we do have the same phenomenon as for Huygens Principle.

1.3 Bernstein-Sato polynomial for Dunkl operators

In the proofs of both cases above (i.e., for $\partial_t^2 - \Delta_k$ and $D_1 \cdots D_n$), it is clear that the very definition of Riesz distributions (the Γ -factor and the exponent), as well as the support structure, hinges on the algebraic identities

$$(\partial_t^2 - \Delta_k)(t^2 - |x|^2)^{s+1} = 4(s+1)(s + \frac{n}{2} + \gamma_k)(t^2 - |x|^2)^s$$

and

$$D_1 \cdots D_n(x_1 \cdots x_n)^{s+1} = \left(\prod_{j=0}^{n-1} (s+1+jk) \right) (x_1 \cdots x_n)^s$$

We may write both identities by the expression

$$P(D)f^{s+1} = b_k(s)f^s$$

where $P(D)$ is obtained from a constant coefficient operator $P(\partial)$ by replacing ∂_i with Dunkl operators D_i , and $b_k(s) \in \mathbb{C}[s]$ may be regarded as the Dunkl version of *Bernstein-Sato polynomial*. If we denote the roots of $b_k(s)$ by $-\beta_i$, i.e.

$$b_k(s) = b_0(s + \beta_1) \cdots (s + \beta_r)$$

we see that the Γ -factor needs to take the form

$$b_0^s \prod_{i=1}^r \Gamma(s - \beta_i + 1)$$

up to some constant (in s), and the support of R_k^s drops precisely when $s = \beta_i - 1$. The phenomenon that more cases of Huygens Principle arise when we put in Dunkl operators can then be attributed to the fact that *Dunkl operators shifts the roots of the Bernstein-Sato polynomial*.

As another prototypical example of Bernstein-Sato polynomial (even though it can not be deformed by Dunkl operators), we cite the result of Gårding [\[Gå47\]](#),

that underlying his generalization of Riesz distributions is the identity³ [Gå48]:

$$\det(\partial_{ij}^*) \det(x_{ij})^{s+1} = (s+1)(s+\frac{3}{2}) \cdots (s+\frac{r+1}{2}) \det(x_{ij})^s$$

where x_{ij} are the standard coordinates on the space of $r \times r$ symmetric matrices (with the understanding that $x_{ij} = x_{ji}$, and the dimension of space is $n = \frac{r(r+1)}{2}$), and

$$\partial_{ij}^* = \varepsilon_{ij} \frac{\partial}{\partial x_{ij}} \quad \varepsilon_{ij} = \begin{cases} 1 & i = j \\ \frac{1}{2} & i \neq j \end{cases}$$

The operator $P(\partial) = \det(\partial_{ij}^*)$ is hyperbolic in the direction of the identity matrix, thus by the general theory (also due to Gårding) admits a fundamental solution supported in the cone \bar{C}_+ of positive semi-definite matrices. By a tour de force of pure analysis, Gårding was able to prove that the Riesz distributions, defined as the analytic continuation of

$$R^s(x) = \frac{1}{c \prod_{i=0}^{r-1} \Gamma(s - \frac{i}{2})} \det(x)^{s - \frac{r+1}{2}} \quad x \in C_+$$

satisfies that $R^0 = \delta$, and for each $m = 0, \frac{1}{2}, 1, \dots, \frac{r-1}{2}$, R^m is supported on the set of positive semi-definite matrices of rank $\leq 2m$, matching exactly the roots of the polynomial (up to a shift)

$$b(s) = (s+1)(s+\frac{3}{2}) \cdots (s+\frac{r+1}{2})$$

In fact, in the lengthy paper of Gårding it is clearly stated that the places where R^s drops support is precisely the poles of the Γ -factor, which in turn is dictated by the polynomial $b(s)$, though in later works this connection is seldom mentioned

³This is a variation of Cayley's identity on the space of *all* $r \times r$ matrices

$$\det(\partial_{ij}) \det(x_{ij})^{s+1} = (s+1)(s+2) \cdots (s+r) \det(x_{ij})^s$$

in which the operator is known as Cayley's Ω -process in classical invariant theory.

explicitly. For instance, Faraut and Koranyi [FK94] sought to determine the set of s for which R^s is a positive measure (see also [So11]). It is true that we can't simply look at the defining expression of R^s or R_k^s , and conclude that the poles of the Γ -factor will make the distribution vanish, but by carefully restricting to an appropriate domain it does work, and it would be closer to Riesz's original approach.

The importance of the polynomial $b(s)$ was not recognized until Bernstein's celebrated result in [Ber71], initiating the theory of (algebraic) \mathcal{D} -modules. He proved by purely algebraic means that, for any polynomial $f \in \mathbb{C}[V]$, there exist a nonzero polynomial $b(s) \in \mathbb{C}[s]$ and a differential operator $P \in \mathcal{D}(V)[s]$ of polynomial coefficients, such that

$$P(s, x, \partial) f^{s+1} = b(s) f^s$$

The set of all such $b(s)$ form an ideal in $\mathbb{C}[s]$, thus is generated by a monic polynomial of minimal degree, called *the Bernstein-Sato polynomial*, or *b-function*, associated with f . The Bernstein-Sato polynomial $b_f(s)$ has been recognized as a very fine invariant of the singularity structure of the algebraic variety defined by f , and is actually difficult to compute in general. A deep result of Kashiwara states that the roots of Bernstein-Sato polynomial all lie in $\mathbb{Q}_{<0}$, which interestingly fails for Dunkl operators, in the few cases that such identities hold.

As remarked before, the relation of the roots of Bernstein-Sato polynomial and the support of the fundamental solution seems to be never explicitly stated. The only result is a rather general (albeit negative) one by Uchida [Uch02], based on the deep results of Atiyah-Bott-Gårding: If the Bernstein-Sato polynomial $b_P(s)$ of $P(\xi)$ has no integer root other than $s = -1$ (e.g., the wave operator for n odd or $n = 2$), then the fundamental solution to $P(\partial)$ has no lacuna.

These Bernstein-Sato identities for Dunkl operators are purely algebraic identities, and they may be of independent interest even if they don't give rise to Riesz distributions. We also study a generalization of the identity for $P(D) = D_1 \cdots D_n$ by using Dunkl operators for complex reflection groups. The classification of complex reflection groups by Shephard-Todd [ST54] consists of an infinite family $G(m, p, n)$ with three integer parameters such that $p \mid m$, and 34 exceptional groups. As special cases, the groups $G(1, 1, n)$, $G(2, 1, n)$ and $G(2, 2, n)$ are precisely Coxeter groups of type A_{n-1} , B_n , and D_n , respectively. We use Dunkl's own expression [DO03] for Dunkl operators with $W = G(m, p, n)$ and parameters $k = (\kappa, k_1, \dots, k_{m/p-1}) \in \mathbb{C}^{m/p}$ (for convenience, let $k_{m/p} = 0$ and the subscript is understood mod m/p):

$$D_i = \frac{\partial}{\partial x_i} + \kappa \sum_{j \neq i} \sum_{l=0}^{m-1} \frac{1 - s_i^{-l} s_{ij} s_i^l}{x_i - \eta^l x_j} + \sum_{r=1}^{m-1} k_r \sum_{l=0}^{m-1} \frac{\eta^{-rl} s_i^l}{x_i}$$

where $\eta = e^{2\pi i/m}$ is a primitive m -th root of unity, and the reflections act by $s_i(x_i) = \eta x_i$, $s_{ij}(x_i) = x_j$ and $s_{ij}(x_j) = x_i$ (and leaving all other coordinates fixed).

We have

Theorem 1.3.1 ($G(m, p, n)$). *For $f(x) = x_1^{m/p} \cdots x_n^{m/p}$, we have*

$$D_1^{m/p} \cdots D_n^{m/p} f^{s+1} = b_k(s) f^s$$

where

$$b_k(s) = \left(\frac{m}{p}\right)^{nm/p} \prod_{j=0}^{n-1} \prod_{r=1}^{m/p} \left(s + \frac{r}{m/p} + j\kappa + k_r\right)$$

We close with a conjecture of Bernstein-Sato polynomials for Dunkl operators

Conjecture 1.3.2. *If the Bernstein-Sato identity*

$$P(\partial) f^{s+1} = b(s) f^s$$

holds with P an operator with constant coefficients, and both P and f are invariant under a complex reflection group $W \subset GL(V)$, then with ∂_i replaced by Dunkl operators D_i for W and any parameters k ,

$$P(D)f^{s+1} = b_k(s)f^s$$

for some polynomial

$$b_k(s) \in \mathbb{C}_k[s] = \mathbb{C}[k][s]$$

that depends on k , such that $b_0(s) = b(s)$.

CHAPTER 2

THE DUNKL WAVE EQUATION

2.1 Wave equation on the half-line

We shall start with the simplest situation ($n = 2$), where we can be fairly explicit without too much technicalities. Consider the Cauchy problem on the half-line with initial and (Dirichlet) boundary conditions

$$\begin{cases} \partial_t^2 u = \left(\partial_x^2 - \frac{k(k+1)}{x^2} \right) u & t \in \mathbb{R}, x > 0 \\ u(0, x) = f(x) & x > 0 \\ \partial_t u(0, x) = g(x) & x > 0 \\ u(t, 0) = 0 & t \in \mathbb{R} \end{cases} \quad (2.1.1)$$

The case of $k = 0$ is a true classic, famously solved by the method of reflection: extend the initial data f and g as odd functions on the whole \mathbb{R} , and solve the new Cauchy problem by the celebrated d'Alembert formula:

$$u(t, x) = \frac{1}{2}(f(x-t) + f(x+t)) + \frac{1}{2} \int_{x-t}^{x+t} g(x') dx'$$

The solution turns out to satisfy the boundary condition at $x = 0$, so must be a solution to (2.1.1) when restricted to $x > 0$.

Note that, to put in modern terms, the fundamental solution

$$E_+(t, x) = \begin{cases} \frac{1}{2} & |x| < t \\ 0 & \text{otherwise} \end{cases}$$

is even in the x variable, so the influence from a source of disturbance at $(0, x_0)$ on the positive half will be exactly cancelled out by the influence from the mirror

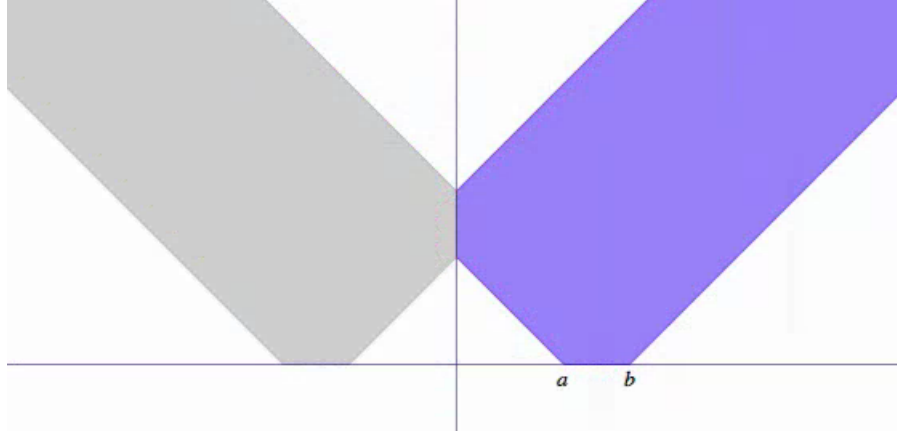


image at $(0, -x_0)$. More precisely, if the initial data f and g are supported on an interval $[a, b]$, then the solution u will vanish for $|t| > x + b$ (in addition to the regions $|t| < a - x$ and $|t| < x - b$ due to the finite speed of propagation, or *hyperbolicity*). In other words, an instantaneous sound will be heard for an extended, but *finite*, duration, and physically speaking the sound was silenced by the reflection off of the “mirror” at $x = 0$. This weaker version of Huygens principle is decidedly a global phenomenon, and can not be detected in the local (i.e., short time) solution near the source.

To study the Cauchy problem for $k > 0$, we introduce the Dunkl operator (on the real line):

$$D = \partial_x + \frac{k}{x}(1 - s)$$

where s is reflection across $x = 0$, i.e. $(s \cdot f)(t, x) = f(t, -x)$. This is a non-local operator, but when restricted to the space of odd (resp. even) functions in x , we can replace $s = -1$ (resp. $s = 1$). So the operator

$$D^2 = \left(\partial_x + \frac{k}{x}(1 - s) \right) \left(\partial_x + \frac{k}{x}(1 - s) \right)$$

when restricted to odd functions, reduces to

$$\begin{aligned}
& \left(\partial_x + \frac{k}{x}(1-s) \right) \left(\partial_x + \frac{2k}{x} \right) \\
&= \left(\partial_x + \frac{0}{x} \right) \left(\partial_x + \frac{2k}{x} \right) \\
&= x^{-k} \left(\partial_x - \frac{k}{x} \right) \left(\partial_x + \frac{k}{x} \right) x^k \\
&= x^{-k} \left(\partial_x^2 - \frac{k(k+1)}{x^2} \right) x^k
\end{aligned}$$

(Here and elsewhere, expressions like x^k are understood to be multiplication operators without further comment.) Therefore, if the Cauchy problem for the *Dunkl wave equation*

$$\begin{cases} \partial_t^2 \tilde{u} = D^2 \tilde{u} & (t, x) \in \mathbb{R} \times \mathbb{R} \\ \tilde{u}(0, x) = \tilde{f}(x) & x \in \mathbb{R} \\ \partial_t \tilde{u}(0, x) = \tilde{g}(x) & x \in \mathbb{R} \end{cases}$$

has a solution which is odd when \tilde{f} and \tilde{g} are odd, then

$$u(t, x) = x^k \tilde{u}(t, x) \quad x > 0$$

solves (2.1.1) with

$$f(x) = x^k \tilde{f}(x) \quad \text{and} \quad g(x) = x^k \tilde{g}(x) \quad x > 0$$

Furthermore, if there is a Huygens Principle for the Dunkl wave equation, then we may deduce a global Huygens Principle for (2.1.1).

2.2 Dunkl operators (algebraic properties)

[We shall refrain from choosing a basis or an inner product, and strive to keep the \mathbb{C} -vector space V separate from its dual V^* (later V will be the complexification of

the space-time \mathbb{R}^n). A likely source of confusion is that x here denotes an element of V^* , regarded as a linear coordinate on V or $V_{\mathbb{R}}$, not a (geometric) point as in $f(x)$ previously and subsequently. Many calculations become more clear with the coordinate-free notations.]

Let $\mathcal{R} \subset V^*$ be a (finite, reduced, non-crystallographic) root system, and \mathcal{R}_+ a choice of positive roots, W the corresponding Coxeter group (finite reflection group) generated by reflections s_α , $\alpha \in \mathcal{R}$:

$$s_\alpha(\xi) = \xi - \langle \alpha, \xi \rangle \alpha^\vee \quad \xi \in V$$

where $\alpha^\vee \in V$ is the coroot of α , normalized by $\langle \alpha, \alpha^\vee \rangle = 2$. Naturally s_α acts on any (reasonable) space of functions on V ; in particular,

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad x \in V^*$$

Note that $\langle s_\alpha(x), s_\alpha(\xi) \rangle = \langle x, \xi \rangle$.

Let $k_\alpha \in \mathbb{C}$ be a parameter for each conjugacy class of s_α in W , or equivalently, the map $\mathcal{R} \rightarrow \mathbb{C}$, $\alpha \mapsto k_\alpha$, is W -invariant, i.e., $k_\alpha = k_{w(\alpha)}$ for all $w \in W$. Define the Dunkl operator $D_\xi = D_\xi(W, k)$ as a deformation of the directional derivative in direction $\xi \in V$:

$$D_\xi := \partial_\xi + \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha \langle \alpha, \xi \rangle}{\alpha} (1 - s_\alpha) \quad (2.2.1)$$

which in particular acts on $\mathbb{C}[V]$ of polynomials by degree -1 . Note that $\xi \mapsto D_\xi$ is linear, and D_ξ is independent of the choice of the positive system \mathcal{R}_+ , nor of the “lengths” of the roots α .

Remark 2.2.1. We may regard D_ξ as an element of $\mathcal{D}(V_{\text{reg}}) \rtimes W$, where $V_{\text{reg}} := V \setminus \bigcup_\alpha H_\alpha$, $H_\alpha := \{\xi \in V \mid \langle \alpha, \xi \rangle = 0\}$. More concretely, we may regard elements of

$\mathbb{C}[V] \cong S(V^*)$, as well as other larger spaces \mathcal{F} , as (\mathbb{C} -valued) functions $\eta \mapsto f(\eta)$ on the vector space V (or a particular real space $V_{\mathbb{R}} \subset V$), and ∂_{ξ} and s_{α} are defined by

$$\partial_{\xi} \cdot f(\eta) = \left. \frac{d}{dt} \right|_{t=0} f(\eta + t\xi) \quad s_{\alpha} \cdot f(\eta) = f(s_{\alpha}^{-1}(\eta))$$

for $f \in \mathcal{F}$. Examples of \mathcal{F} include $C^{\infty}(V, \mathbb{C})$, $\mathcal{S}'(V_{\mathbb{R}})$, and $\mathbb{C}[V_{\text{reg}}] = \mathbb{C}[V][\alpha^{-1}, \alpha \in \mathcal{R}_+]$, so long as ∂_{ξ} and $\frac{1-s_{\alpha}}{\alpha}$ preserve the space \mathcal{F} . We shall leave $\mathcal{S}'(V_{\mathbb{R}})$ until the next section, as it requires special care.

Theorem 2.2.1 (Dunkl). *The Dunkl operators, for fixed W and $k : \mathcal{R} \rightarrow \mathbb{C}$, satisfy the following relations*

1. $D_{\xi}D_{\eta} = D_{\eta}D_{\xi}$ (commutativity)
2. $wD_{\xi} = D_{w(\xi)}w$ (W -equivariance)
3. $[D_{\xi}, x] = \langle x, \xi \rangle + \sum_{\alpha \in \mathcal{R}_+} k_{\alpha} \langle x, \alpha^{\vee} \rangle \langle \alpha, \xi \rangle s_{\alpha}$

for all $\xi, \eta \in V$, $x \in V^*$ and $w \in W$.

Remark 2.2.2. These relations, along with $xy = yx$ for all $x, y \in V^*$ and $s_{\alpha}x = s_{\alpha}(x)s_{\alpha}^{-1}$ ¹, are the defining relations of the *rational Cherednik algebra* $H_k(W)$, after the work of Etingof-Ginzburg [EG01], and the space $\mathbb{C}[V]$, or more generally \mathcal{F} , becomes a left $H_k(W)$ -module. As a vector space,

$$H_k(W) \cong \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \quad (\text{Poincaré-Birkhoff-Witt property})$$

and it may be more suggestive to write

$$H_k(W) \cong \mathbb{C}[V] \rtimes \mathbb{C}W \rtimes \mathbb{C}[V^*]$$

¹Note that $s_{\alpha}x$ is the composition of two operators (on \mathcal{F}), while $s_{\alpha}(x)$ is an element of V^* .

to indicate the algebra structure coming from the natural actions of W on the other two factors. There also is a generalization for any complex reflection group $W \subset GL(V)$ [DO03], see §3.3.

Proof. The identities (2) and (3) follow easily from the definition:

$$\begin{aligned}
s_\beta D_\xi &= s_\beta \partial_\xi + \sum_{\alpha \in \mathcal{R}_+} s_\beta \frac{k_\alpha \langle \alpha, \xi \rangle}{\alpha} (1 - s_\alpha) \\
&= \partial_{s_\beta(\xi)} s_\beta + \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha \langle \alpha, \xi \rangle}{s_\beta(\alpha)} s_\beta (1 - s_\alpha) \\
&= \partial_{s_\beta(\xi)} s_\beta + \sum_{s_\beta(\alpha) \in s_\beta(\mathcal{R}_+)} \frac{k_\alpha \langle s_\beta(\alpha), s_\beta(\xi) \rangle}{s_\beta(\alpha)} (1 - s_{s_\beta(\alpha)}) s_\beta = D_{s_\beta(\xi)} s_\beta
\end{aligned}$$

(which uses the W -invariance of k)

$$\begin{aligned}
[D_\xi, x] &= [\partial_\xi, x] + \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha \langle \alpha, \xi \rangle}{\alpha} [1 - s_\alpha, x] \\
&= \langle x, \xi \rangle + \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha \langle \alpha, \xi \rangle}{\alpha} (x s_\alpha - s_\alpha x) \\
&= \langle x, \xi \rangle + \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha \langle \alpha, \xi \rangle}{\alpha} (x s_\alpha - (x - \langle x, \alpha^\vee \rangle \alpha) s_\alpha) \\
&= \langle x, \xi \rangle + \sum_{\alpha \in \mathcal{R}_+} k_\alpha \langle x, \alpha^\vee \rangle \langle \alpha, \xi \rangle s_\alpha
\end{aligned}$$

while (1) was a difficult computation by Dunkl [Du89], also c.f. [He91], but in fact it could be deduced via an easy argument in [Et07] (p. ?). For any $\xi, \eta \in V$ and $x \in V^*$, consider

$$[[D_\xi, D_\eta], x] = [[D_\xi, x], D_\eta] - [[D_\eta, x], D_\xi]$$

and using (2) and (3), the first term becomes

$$\begin{aligned}
[[D_\xi, x], D_\eta] &= [\langle x, \xi \rangle + \sum_{\alpha} k_{\alpha} \langle x, \alpha^{\vee} \rangle \langle \alpha, \xi \rangle s_{\alpha}, D_\eta] \\
&= \sum_{\alpha} k_{\alpha} \langle x, \alpha^{\vee} \rangle \langle \alpha, \xi \rangle [s_{\alpha}, D_\eta] \\
&= \sum_{\alpha} k_{\alpha} \langle x, \alpha^{\vee} \rangle \langle \alpha, \xi \rangle s_{\alpha} \langle \alpha, \eta \rangle D_{\alpha^{\vee}}
\end{aligned}$$

Since ξ and η appear symmetrically it follows that $[[D_\xi, D_\eta], x] = 0$ for any $x \in V^*$.

This means that for any $f \in \mathbb{C}[V]$,

$$[D_\xi, D_\eta] \cdot f = f[D_\xi, D_\eta] \cdot 1 = 0$$

As $\mathbb{C}[V]$ is a faithful representation of $H_k(W)$, it follows that $[D_\xi, D_\eta] = 0$. \square

Example (Type A_{n-1} , $n \geq 2$). Let $V = \mathbb{C}^n$, and $W = S_n$ acting on V and V^* by permuting the standard bases $\{\xi_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^n$, satisfying $\langle x_i, \xi_j \rangle = \delta_{ij}$. Then $\mathcal{R} = \{x_i - x_j\}_{i \neq j}$ and $k_{\alpha} = k$ (all reflections/transpositions are conjugate), and the Dunkl operator (in the direction of ξ_i) takes the form

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j \neq i} \frac{k}{x_i - x_j} (1 - s_{ij}).$$

Even in this case, the commutativity $D_i D_j = D_j D_i$ is non-trivial.

It is well-known that the irreducible root systems are classified up to scalings by Coxeter (for classifying regular polytopes), and the standard nomenclature (largely due to E. Cartan) consists of types A_{n-1} , $B_n = C_n$ ($n \geq 2$), D_n ($n \geq 4$), E_6 , E_7 , E_8 , F_4 , $G_2 = I_2(6)$, H_3 , H_4 , and $I_2(m)$ ($m = 5, m \geq 7$), where the subscript, known as the *rank*, is the dimension of the span of all the roots. The types A through G are the *crystallographic* root systems in the sense that, after a particular choice of scalings, the roots generate a lattice of the same rank (or equivalently, for any two roots α and β , $s_{\alpha}(\beta) = \beta - r\alpha$ for some $r \in \mathbb{N}$), and the

corresponding Coxeter groups are precisely the Weyl groups of finite-dimensional simple Lie algebras and associated Lie groups (hence our notation W). The types $E_{6,7,8}$, F_4 and G_2 , along with H_3 and H_4 , are called exceptional Coxeter groups, whereas the types $I_2(m)$ are simply dihedral groups of order $2m$, and we have the identification $I_2(2) = A_1 \times A_1$, $I_2(3) = A_2$, $I_2(4) = B_2$, $I_2(6) = G_2$.

Example (Type $B_n = C_n$, $n \geq 2$). Let $V = \mathbb{C}^n$, and $W = S_n \ltimes (\mathbb{Z}_2)^n$. $\mathcal{R} = \{\pm x_i \pm x_j\}_{i \neq j} \cup \{\pm x_i\}_{i=1}^n$, consisting of two W -orbits. Denote $s_i = s_{x_i}$, $s_{ij} = s_{x_i - x_j}$, and $r_{ij} = s_{x_i + x_j}$, or in terms of their action on V^*

$$s_i(x_i) = -x_i$$

$$s_{ij}(x_i) = x_j \quad s_{ij}(x_j) = x_i$$

$$r_{ij}(x_i) = -x_j \quad r_{ij}(x_j) = -x_i$$

leaving all other coordinates fixed, then

$$D_i = \frac{\partial}{\partial x_i} + \frac{k_1}{x_i} + \sum_{j \neq i} \frac{k_2}{x_i - x_j} (1 - s_{ij}) + \frac{k_2}{x_i + x_j} (1 - r_{ij})$$

The commutativity of Dunkl operators is crucial, as it enables one to replace ordinary derivatives ∂_ξ by D_ξ and expect that, with some work, many notions and theorems from classical analysis have a counterpart for Dunkl operators (at least for generic k). One of the most important objects is the Dunkl Laplacian: choose a basis $\{\xi_i\}_{i=1}^n$ for V such that the induced bilinear form $(x|y) := \sum_{i=1}^n \langle x, \xi_i \rangle \langle y, \xi_i \rangle$ is W -invariant², and define (writing $D_i = D_{\xi_i}$ for short)

$$\Delta_k := \sum_{i=1}^n D_i^2 = \Delta + \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha(\alpha|\alpha)}{\alpha} \left(\partial_{\alpha^\vee} - \frac{1 - s_\alpha}{\alpha} \right) \quad (2.2.2)$$

²The existence can be shown by first defining the bilinear form as $(x|y) = \sum_{\alpha \in \mathcal{R}_+} \langle x, \alpha^\vee \rangle \langle y, \alpha^\vee \rangle$, which obviously is W -invariant, and a basis $\{\xi_i\}$ then follows from it.

which was the original motivation for Dunkl in his study of orthogonal polynomials. Soon after Dunkl's original work, Heckman made the following simple yet important observation that reduced the (quantum) integrability of the Calogero-Moser system to an easy consequence of Dunkl's commutativity (see the remark below). Recall that the Calogero-Moser operator associated with a root system \mathcal{R} and parameter k is defined by

$$L_k := \Delta - \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha(k_\alpha + 1)(\alpha|\alpha)}{\alpha^2} \quad (2.2.3)$$

(The original Calogero-Moser system was of type A , and was generalized to all root systems by Olshanetsky-Perelomov.)

Lemma 2.2.2 (Heckman). *Let $\varepsilon : W \rightarrow \{\pm 1\}$ be a (linear) character. Then the Dunkl Laplacian Δ_k , when restricted to the space \mathcal{F}^ε of ε -relative invariant functions in \mathcal{F} (i.e., $f \in \mathcal{F}$ such that $s_\alpha \cdot f = \varepsilon(s_\alpha)f$ for all $\alpha \in \mathcal{R}_+$), coincides with a Calogero-Moser operator up to conjugation. More precisely,*

$$\Delta_k \cdot f = \Theta^{-1} L_{k^\varepsilon} \Theta \cdot f \quad \text{for all } f \in \mathcal{F}^\varepsilon$$

where $\Theta \equiv \Theta_k := \prod_{\alpha \in \mathcal{R}_+} \alpha^{k_\alpha}$ and $k^\varepsilon : \mathcal{R} \rightarrow \mathbb{C}$ is given by

$$k_\alpha^\varepsilon := \begin{cases} k_\alpha & \text{if } \varepsilon(s_\alpha) = -1 \\ k_\alpha - 1 & \text{if } \varepsilon(s_\alpha) = 1 \end{cases}$$

Remark 2.2.3. The function Θ is multi-valued for non-integral k , yet $\Theta^{-1} L_k \Theta$ is a well-defined differential operator with rational coefficients. In other words, conjugation by Θ is an (outer) automorphism on $\mathcal{D}(V_{\text{reg}}) \cong \mathbb{C}[V_{\text{reg}}] \otimes \mathbb{C}[V^*]$.

Remark 2.2.4. In terms of rational Cherednik algebra, the restriction process of Heckman is formalized as an algebra homomorphism $eH_k e \rightarrow \mathcal{D}(V_{\text{reg}})^W$ from the spherical subalgebra to W -invariant differential operators with coefficients in $\mathbb{C}[V_{\text{reg}}]$.

Remark 2.2.5. By quantum integrability of a differential operator L on \mathbb{R}^n is meant the existence of n (algebraically independent) differential operators, one of which being L , that all commute with each other. For the Calogero-Moser operator L_k , the n operators can be given explicitly as the restriction of $P_1(D_i), \dots, P_n(D_i)$ on the space \mathcal{F}^W of W -invariant functions, where P_1, \dots, P_n are the (free) generators of $\mathbb{C}[V^*]^W$ (Chevalley's theorem) and $D_i = D_i(k+1)$ are Dunkl operators with parameters $k_\alpha + 1$. Remarkably still, for $k_\alpha \in \mathbb{N}$, there exists an additional operator that commutes with each of the n operators. In other words, the commutative algebra generated by those n differential operators is not maximal, and in that case L_k is said to be *algebraically integrable* after Chalykh-Veselov [CV90]. For $n = 1$, this goes back to the work of Burchnell-Chaundy, and was later revived by the work of Krichever. For more, see the survey [Ch08] and references therein.

Proof. Although it would follow from the formula (2.2.2) and the identity

$$\Theta^{-1} \Delta \Theta = \Delta + \sum_{\alpha} \frac{k_{\alpha}(\alpha|\alpha)}{\alpha} + \sum_{\alpha} \frac{(k_{\alpha}^2 - k_{\alpha})(\alpha|\alpha)}{\alpha^2} \quad (2.2.4)$$

we shall provide a direct calculation. Denote by Res^{ε} the process of restricting to $\mathcal{F}^{\varepsilon}$, i.e. for $P = P(D)$, $\text{Res}^{\varepsilon} P$ is the (unique) differential operator such that

$$P \cdot f = \text{Res}^{\varepsilon} P \cdot f \quad \text{for all } f \in \mathcal{F}^{\varepsilon}$$

In particular

$$D_{\xi}^{\varepsilon} := \text{Res}^{\varepsilon} D_{\xi} = \partial_{\xi} + \sum_{\alpha} \frac{k_{\alpha} \langle \alpha, \xi \rangle}{\alpha} (1 - \varepsilon(s_{\alpha}))$$

which also satisfies linearity and W -equivariance of the map $\xi \mapsto D_{\xi}^{\varepsilon}$ ($w D_{\xi}^{\varepsilon} =$

$D_{w(\xi)}^\varepsilon w$ for all $w \in W$). Now, for each i ,

$$\begin{aligned}
\text{Res}^\varepsilon D_i D_i &= \left(\partial_i + \sum_{\alpha} \frac{k_{\alpha} \langle \alpha, \xi_i \rangle}{\alpha} (1 - s_{\alpha}) \right) D_i^\varepsilon \\
&= \partial_i D_i^\varepsilon + \sum_{\alpha} \frac{k_{\alpha} \langle \alpha, \xi_i \rangle}{\alpha} (D_i^\varepsilon - D_{s_{\alpha}(\xi_i)}^\varepsilon s_{\alpha}) \\
&= \partial_i D_i^\varepsilon + \sum_{\alpha} \frac{k_{\alpha} \langle \alpha, \xi_i \rangle}{\alpha} (D_i^\varepsilon - D_{\xi_i - \langle \alpha, \xi_i \rangle \alpha^\vee}^\varepsilon \varepsilon(s_{\alpha})) \\
&= \left(\partial_i + \sum_{\alpha} \frac{k_{\alpha} \langle \alpha, \xi_i \rangle}{\alpha} (1 - \varepsilon(s_{\alpha})) \right) D_i^\varepsilon + \sum_{\alpha} \frac{k_{\alpha} \langle \alpha, \xi_i \rangle^2}{\alpha} D_{\alpha^\vee}^\varepsilon \varepsilon(s_{\alpha}) \\
&= D_i^\varepsilon D_i^\varepsilon + \sum_{\alpha} \frac{k_{\alpha} \langle \alpha, \xi_i \rangle^2}{\alpha} \varepsilon(s_{\alpha}) D_{\alpha^\vee}^\varepsilon
\end{aligned}$$

Summing over i , we get

$$\begin{aligned}
\text{Res}^\varepsilon \Delta_k &= \sum_{i=1}^n (D_i^\varepsilon)^2 + \sum_{\alpha} \frac{\varepsilon(s_{\alpha}) k_{\alpha} (\alpha|\alpha)}{\alpha} \left(\partial_{\alpha^\vee} + \sum_{\beta} \frac{k_{\beta} \langle \beta, \alpha^\vee \rangle}{\beta} (1 - \varepsilon(s_{\beta})) \right) \\
&= \Delta + \sum_{\alpha} \frac{k_{\alpha} (\alpha|\alpha)}{\alpha} \partial_{\alpha^\vee} - \sum_{\alpha} \frac{k_{\alpha} (\alpha|\alpha)}{\alpha^2} (1 - \varepsilon(s_{\alpha}))
\end{aligned}$$

where we repeatedly used the identity

$$\sum_{i=1}^n \langle \alpha, \xi_i \rangle \xi_i = \frac{(\alpha|\alpha)}{2} \alpha^\vee.$$

Proof of identity. We shall show that the left hand side is an eigenvector of s_{α} with eigenvalue -1 , hence is a multiple of α^\vee . Indeed, by pairing with $x \in V^*$,

$$\begin{aligned}
\langle x, s_{\alpha}(\sum_{i=1}^n \langle \alpha, \xi_i \rangle \xi_i) \rangle &= \sum_{i=1}^n \langle \alpha, \xi_i \rangle \langle x, s_{\alpha}(\xi_i) \rangle \\
&= \sum_{i=1}^n \langle \alpha, \xi_i \rangle \langle s_{\alpha}(x), \xi_i \rangle \\
&= (\alpha|s_{\alpha}(x)) \\
&= (s_{\alpha}(\alpha)|x) \\
&= (-\alpha|x) \\
&= \langle x, -\sum_{i=1}^n \langle \alpha, \xi_i \rangle \xi_i \rangle
\end{aligned}$$

Therefore,

$$\sum_{i=1}^n \langle \alpha, \xi_i \rangle \xi_i = c \alpha^\vee$$

for some $c \in \mathbb{C}$. To determine c , we shall pair both sides with α :

$$(\alpha|\alpha) = c \langle \alpha, \alpha^\vee \rangle = 2c$$

which proves the identity. □

□

Let $\{x_i\}_{i=1}^n$ be the dual basis of $\{\xi_i\}_{i=1}^n$, i.e., $\langle x_i, \xi_j \rangle = \delta_{ij}$, and denote

$$|x|^2 := \sum_{i=1}^n x_i^2 \in \mathbb{C}[V]$$

Lemma 2.2.3 (Heckman). *The operators*

$$e_+ = \frac{1}{2}|x|^2 \quad e_- = -\frac{1}{2}\Delta_k \quad h = \frac{1}{2} \sum_{i=1}^n (x_i D_i + D_i x_i)$$

satisfy the relations of \mathfrak{sl}_2 triple:

$$[h, e_+] = 2e_+ \quad [h, e_-] = -2e_- \quad [e_+, e_-] = h$$

Remark 2.2.6. These operators may (and should) be regarded as elements of $H_k(W)$, in which the relations $[h, x] = x$ and $[h, \xi] = -\xi$ hold for all $x \in V^*$ and $\xi \in V$.

Remark 2.2.7. Ben Saïd [BS07] showed that, for $k > 0$, this representation of $\mathfrak{sl}_2(\mathbb{R})$ exponentiates to a representation of the universal cover of $SL_2(\mathbb{R})$ on $L^2(\mathbb{R}^n, v_k d\mu)$, which in particular includes the Dunkl transform as $\exp\left(\frac{i\pi}{4}(e_+ + e_-)\right)$. Moreover, if $\gamma_k + \frac{n}{2} \in \mathbb{N}$ (resp. $\in \frac{\mathbb{N}}{2}$), it factors through a representation of $SL_2(\mathbb{R})$ itself (resp. the metaplectic group $\widetilde{SL}_2(\mathbb{R})$, the double cover of $SL_2(\mathbb{R})$).

Proof. First note that h is simply the Euler operator up to a shift. More precisely,

$$h = \sum_{i=1}^n x_i \partial_i + \frac{n}{2} + \gamma_k \quad (2.2.5)$$

where $\gamma_k := \sum_{\alpha \in \mathcal{R}_+} k_\alpha$. Indeed, using the fact that $\sum_i \langle \alpha, \xi_i \rangle x_i = \alpha$, we have

$$\begin{aligned} \sum_i x_i D_i &= \sum_i x_i \left(\partial_i + \sum_\alpha \frac{k_\alpha \langle \alpha, \xi_i \rangle}{\alpha} (1 - s_\alpha) \right) \\ &= \sum_i x_i \partial_i + \sum_\alpha k_\alpha (1 - s_\alpha) \end{aligned}$$

and similarly

$$\begin{aligned} \sum_i D_i x_i &= \sum_i \partial_i x_i + \sum_\alpha \frac{k_\alpha}{\alpha} (1 - s_\alpha) \alpha \\ &= \sum_i (x_i \partial_i + 1) + \sum_\alpha k_\alpha (1 + s_\alpha) \end{aligned}$$

since $s_\alpha \alpha = -\alpha s_\alpha$, and (2.2.5) follows.

Now, $[h, e_\pm] = \pm 2e_\pm$ simply states the fact that e_+ increases (resp. e_- decreases) the order of homogeneity by 2. For the relation $[e_+, e_-] = h$, we compute

$$[D_i^2, x_j^2] = D_i x_j [D_i, x_j] + D_i [D_i, x_j] x_j + x_j [D_i, x_j] D_i + [D_i, x_j] x_j D_i$$

where

$$\begin{aligned} [D_i, x_j] &= \langle x_j, \xi_i \rangle + \sum_\alpha k_\alpha \langle x_j, \alpha^\vee \rangle \langle \alpha, \xi_i \rangle s_\alpha \\ &= \delta_{ij} + \sum_\alpha k_\alpha \langle \alpha, \xi_j \rangle \langle x_i, \alpha^\vee \rangle s_\alpha \end{aligned}$$

When summed up over i and j , we see that

$$\begin{aligned}
[e_+, e_-] &= \frac{1}{4} \sum_{i,j} [D_i^2, x_j^2] \\
&= \frac{1}{4} \sum_i (2D_i x_i + 2x_i D_i) \\
&\quad + \frac{1}{4} \sum_{\alpha} k_{\alpha} (D_{\alpha^{\vee}} \alpha s_{\alpha} + D_{\alpha^{\vee}} s_{\alpha} \alpha + \alpha s_{\alpha} D_{\alpha^{\vee}} + s_{\alpha} \alpha D_{\alpha^{\vee}}) \\
&= \frac{1}{2} \sum_i (x_i D_i + D_i x_i) = h
\end{aligned}$$

again due to $\alpha s_{\alpha} = -s_{\alpha} \alpha$. □

As a last remark, this Lemma still holds if we take e_+ to be any (non-degenerate) quadratic element in $\mathbb{C}[V]^W$ and take $e_- \in \mathbb{C}[V^*]^W$ to be its (formal) Fourier dual. If one direction is fixed by W , e_- may be regarded as the Dunkl wave operator when restricted to a real space $V_{\mathbb{R}}$ for which the signature is $(+, -, \dots, -)$. In that case, it is customary to use t (for time) as the coordinate in the $+$ direction, and the Dunkl wave operator shall be denoted $\partial_t^2 - \Delta_k$, where Δ_k is the usual Dunkl Laplacian on the $(n-1)$ -dimensional W -invariant complement. One important example is the second elementary symmetric polynomial

$$\sigma_2(x) = \sum_{i < j} x_i x_j \in \mathbb{C}[V]^W$$

where $W = S_n$ acting on $V \cong \mathbb{C}^n$. The direction that W fixes is $(1, \dots, 1)$, and as a quadratic form on $V_{\mathbb{R}} = \mathbb{R}^n$, the signature of σ_2 is indeed $(+, -, \dots, -)$, hence with a change of variables the operator e_- is of the form $\partial_t^2 - \Delta_k$.

2.3 Dunkl operators acting on distributions

We wish to define the Dunkl version of Riesz distributions $R_k^s \in \mathcal{S}'(\mathbb{R}^n)$, $s \in \mathbb{C}$, so that $(\partial_t^2 - \Delta_k) R_k^s = R_k^{s-1}$ and $R_k^0 = \delta$. The first task is to discuss how Dunkl

operators act on distributions in general. It should be noted at the outset that the space of (tempered) distributions $\mathcal{S}'(\mathbb{R}^n)$ remains the same; it is the natural embedding of (locally integrable) functions into distribution that is being modified. Also, the use of *tempered* distributions is not very important.

Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution, and denote the pairing with $\phi \in \mathcal{S}(\mathbb{R}^n)$ by either of the two notations

$$\langle u, \phi \rangle = \int u(x)\phi(x)dx \quad (\text{formally})$$

whichever is convenient. Of central importance is $\delta \in \mathcal{S}'(\mathbb{R}^n)$, defined by

$$\langle \delta, \phi \rangle = \phi(0) \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$$

We shall define the distribution $D_i u$ by

$$\langle D_i u, \phi \rangle = \langle u, -D_i \phi \rangle$$

which is equivalent to defining

$$D_i u = \partial_i u - \sum_{\alpha \in \mathcal{R}_+} \frac{k_\alpha \langle \alpha, \xi_i \rangle}{\alpha} (1 - s_\alpha) u$$

with the usual meanings of $\partial_i u$ and $s_\alpha u$:

$$\langle \partial_i u, \phi \rangle = \langle u, -\partial_i \phi \rangle \quad \langle s_\alpha u, \phi \rangle = \langle u, s_\alpha \phi \rangle$$

Note the sign difference from expression (2.2.1).

The following proposition justifies our definition.

Proposition 2.3.1. (1) *The space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions becomes a left module over $H_k(W)$, with $\xi_i \in V$ acting as the Dunkl operator D_i .*

(2) For $\operatorname{Re} k_\alpha > -\frac{1}{2}$, the embedding $f \mapsto \mathcal{T}_k(f) \in \mathcal{S}'(\mathbb{R}^n)$

$$\langle \mathcal{T}_k(f), \phi \rangle = \int_{\mathbb{R}^n} f(x) \phi(x) v_k(x) dx, \quad v_k(x) := \prod_{\alpha \in \mathcal{R}_+} |\alpha(x)|^{2k_\alpha}$$

of (piecewise) C^1 functions of polynomial growth (at infinity) into tempered distributions commutes with Dunkl operators:

$$D_i \mathcal{T}_k(f) = \mathcal{T}_k(D_i f)$$

In other words, \mathcal{T}_k intertwines with the $H_k(W)$ action.

Proof. One has to be cautious of *not* using the relations of Dunkl operators in §2.2 before passing them over to $\phi \in \mathcal{S}(\mathbb{R}^n)$. For the commutator $[D_i, x_j]$,

$$\begin{aligned} \langle [D_i, x_j] u, \phi \rangle &= \langle u, [x_j, -D_i] \phi \rangle \\ &= \langle u, \left(\langle x_j, \xi_i \rangle + \sum_{\alpha} k_{\alpha} \langle x_j, \alpha^\vee \rangle \langle \alpha, \xi_i \rangle s_{\alpha} \right) \phi \rangle \\ &= \langle \left(\langle x_j, \xi_i \rangle + \sum_{\alpha} k_{\alpha} \langle x_j, \alpha^\vee \rangle \langle \alpha, \xi_i \rangle s_{\alpha} \right) u, \phi \rangle \end{aligned}$$

and similar manipulations show that these operators generate the same algebra $H_k(W)$, or in other words, $H_k(W)$ acts on distributions (from the left).

To show (2), we need to prove the Dunkl version of integration by parts:

$$- \int f(x) D_i \phi(x) v_k(x) dx = \int D_i f(x) \phi(x) v_k(x) dx \quad (2.3.1)$$

or

$$\int (f(x) D_i \phi(x) + D_i f(x) \phi(x)) v_k(x) dx = 0$$

for all possible f and $\phi \in \mathcal{S}(\mathbb{R}^n)$. The differential part gives

$$\begin{aligned} \int (f(x) \partial_i \phi(x) + \partial_i f(x) \phi(x)) v_k(x) dx &= \int \partial_i (f(x) \phi(x)) v_k(x) dx \\ &= - \int f(x) \phi(x) \partial_i v_k(x) dx \\ &= - \int f(x) \phi(x) \left(\sum_{\alpha} \frac{2k_{\alpha} \langle \alpha, \xi_i \rangle}{\alpha(x)} \right) v_k(x) dx \end{aligned}$$

while the reflection part yields

$$\begin{aligned}
& \int \sum_{\alpha} \frac{k_{\alpha} \langle \alpha, \xi_i \rangle}{\alpha(x)} \left(2f(x)\phi(x) - f(s_{\alpha}x)\phi(x) - f(x)\phi(s_{\alpha}x) \right) v_k(x) dx \\
&= \int \sum_{\alpha} \frac{2k_{\alpha} \langle \alpha, \xi_i \rangle}{\alpha(x)} f(x)\phi(x) v_k(x) dx \\
&\quad - \sum_{\alpha} k_{\alpha} \langle \alpha, \xi_i \rangle \int \frac{f(s_{\alpha}x)\phi(x) + f(x)\phi(s_{\alpha}x)}{\alpha(x)} v_k(x) dx
\end{aligned}$$

where the latter integral vanishes by symmetry across the $\alpha(x) = 0$ plane (or more formally by a change of variables $y = s_{\alpha}x$ and note that $v_k(y) = v_k(x)$ and $\alpha(y) = -\alpha(x)$). The claim (2.3.1) follows.

A minor technical detail is that the last integral by itself might not converge at $\alpha(x) = 0$ if k is small, which may be circumvented by first assuming $\operatorname{Re} k > 1$ and analytically continue to all $\operatorname{Re} k > -\frac{1}{2}$, or more careful with not splitting the integrals. \square

In particular, the \mathfrak{sl}_2 triple in $H_k(W)$

$$e_+ = \frac{1}{2} \sum_{i=1}^n x_i^2 \quad e_- = -\frac{1}{2} \sum_{i=1}^n D_i^2 \quad h = \frac{1}{2} \sum_{i=1}^n (x_i D_i + D_i x_i)$$

act on distributions. For example, if f is a locally integrable function on \mathbb{R}^n which is homogeneous of degree $s \in \mathbb{C}$ (i.e., $f(tx) = t^s f(x)$ for all $t \in \mathbb{R}_{>0}$ and all $x \in \mathbb{R}^n$), then

$$\begin{aligned}
h \cdot \mathcal{T}_k(f) &= \mathcal{T}_k(h \cdot f) \\
&= \mathcal{T}_k\left(\left(s + \frac{n}{2} + \gamma_k\right)f\right) \\
&= \left(s + \frac{n}{2} + \gamma_k\right) \mathcal{T}_k(f)
\end{aligned} \tag{2.3.2}$$

whereas,

$$h \cdot \delta = -\left(\frac{n}{2} + \gamma_k\right) \delta \tag{2.3.3}$$

since

$$\begin{aligned}
\langle h \cdot \delta, \phi \rangle &= \left\langle \frac{1}{2} \sum_i (x_i D_i + D_i x_i) \delta, \phi \right\rangle \\
&= \left\langle \delta, -\frac{1}{2} \sum_i (D_i x_i + x_i D_i) \phi \right\rangle \\
&= \left\langle \delta, -\left(\sum_i x_i \partial_i + \frac{n}{2} + \gamma_k \right) \phi \right\rangle \\
&= -\left(\frac{n}{2} + \gamma_k \right) \phi(0)
\end{aligned}$$

Note that we could not use the identity (2.2.5) before passing over to f or ϕ . Note also that the difference in the order of “homogeneity” between constant functions and δ is widened by $2\gamma_k$. Intuitively speaking, as we shall see again, *the introduction of parameters $k > 0$ has the same effect as increasing the dimension of space-time by $2\gamma_k$, thereby admitting more cases of Huygens Principle.*

One thing that is different from ordinary derivatives is that D_i is no longer a local operator, i.e. $\text{supp } D_i u \not\subseteq \text{supp } u$ in general. Recall that the support of a distribution u is by definition the complement of the largest open set $U \subseteq \mathbb{R}^n$ such that $\langle u, \phi \rangle = 0$ for all ϕ supported in U . It’s easy to see that, if $\text{supp } u$ is W -invariant (as will be the case for Riesz distributions), then $\text{supp } D_i u \subseteq \text{supp } u$ does hold.

We shall also make use of two basic operations of distributions: pullback along a submersion, and tensor product (also known as direct product) of two distributions.

We note in passing that a direct modification of the classical approach to distributions supported on a manifold of lower dimension, as in Gel’fand-Shilov ([GS58] Vol. I, Chapter III), leads quickly to a construction of the fundamental solution to

the Dunkl wave equation without Riesz distributions, though it falls short of being a complete proof. Let P be a differentiable function on \mathbb{R}^n that defines a *smooth* hypersurface $\{x \in \mathbb{R}^n \mid P(x) = 0\}$. Then $\delta(P)$, as well as $\delta'(P), \dots, \delta^{(\ell)}(P), \dots$, can be defined as the pullback of $\delta, \delta', \dots \in \mathcal{D}'(\mathbb{R}^1)$ along $P : \mathbb{R}^n \rightarrow \mathbb{R}^1$. More directly, we follow the approach in Gelf'and-Shilov.

Let the Gelf'and-Leray form ω be an $(n-1)$ -form on \mathbb{R}^n such that $dP \wedge \omega = dx_1 \cdots dx_n$, the volume form. Then

$$\langle \delta(P), \phi \rangle = \int_{P=0} \phi(x) \omega$$

and for $\delta^{(\ell)}(P)$, for $\ell \geq 1$, define the $(n-1)$ -form $\omega_\ell(\phi)$ for a fixed $\phi \in \mathcal{S}(\mathbb{R}^n)$ recursively

$$\omega_0(\phi) = \phi \omega$$

$$d\omega_0(\phi) = dP \wedge \omega_1(\phi)$$

$$\dots$$

$$d\omega_{\ell-1}(\phi) = dP \wedge \omega_\ell(\phi)$$

and define

$$\langle \delta^{(\ell)}(P), \phi \rangle = (-1)^\ell \int_{P=0} \omega_\ell(\phi)$$

which is independent of the choices made in defining $\omega_0(\phi), \dots, \omega_\ell(\phi)$. One can then justify the identities obtained from formally differentiating $P\delta(P) = 0$ in the “variable” P :

$$P\delta'(P) + \delta(P) = 0$$

$$P\delta''(P) + 2\delta'(P) = 0$$

$$\dots$$

$$P\delta^{(\ell)}(P) + \ell\delta^{(\ell-1)}(P) = 0$$

So far nothing is new. To let Dunkl operators act on these distributions, we need P to be W -invariant. Then the distributions $\delta^{(\ell)}(P)$ are W -invariant, and we have the chain rule:

$$D_i \delta^{(\ell)}(P) = \delta^{(\ell+1)}(P) D_i P$$

Example. With W acting on \mathbb{R}^{n-1} and $P = t^2 - |x|^2 + \epsilon$, for real $\epsilon \neq 0$, we have

$$D_i \delta^{(\ell)}(P) = \delta^{(\ell+1)}(P) (-2x_i)$$

and

$$\begin{aligned} D_i^2 \delta^{(\ell)}(P) &= \delta^{(\ell+2)}(P) (4x_i^2) - 2\delta^{(\ell+1)}(P) (D_i x_i) \\ &= \delta^{(\ell+2)}(P) (4x_i^2) - 2\delta^{(\ell+1)}(P) \left(1 + \sum_{\alpha \in \mathcal{R}_+} k_\alpha \langle x_i, \alpha^\vee \rangle \langle \alpha, \xi_i \rangle \right) \end{aligned}$$

Summing over i , we see that $\sum_i \langle \alpha, \xi_i \rangle \langle x_i, \alpha^\vee \rangle = \langle \alpha, \alpha^\vee \rangle = 2$, and consequently

$$\begin{aligned} (\partial_t^2 - \Delta_k) \delta^{(\ell)}(P) &= 4(P - \epsilon) \delta^{(\ell+2)}(P) + 2(n + 2\gamma_k) \delta^{(\ell+1)}(P) \\ &= (2n + 4\gamma_k - 4(\ell + 2)) \delta^{(\ell+1)}(P) - 4\epsilon \delta^{(\ell+2)}(P) \end{aligned}$$

using the identity $P \delta^{(\ell+2)}(P) = -(\ell + 2) \delta^{(\ell+1)}(P)$. If ℓ is just the right value (namely $\ell = \frac{n}{2} + \gamma_k - 2$, only possible when $n + 2\gamma_k \geq 4$ and even), then the first term vanishes. Letting $\epsilon \rightarrow 0$ makes $\delta^{(\ell)}(P)$ a distributional solution to the wave equation.

The difficulty is to justify the calculation for $\epsilon = 0$, for which $P = 0$ is singular (at the origin). To that end, analytic continuation of Riesz distributions comes to play.

2.4 Riesz distributions

For the Riesz distributions to work for Dunkl wave operator, it is important to have the Bernstein-Sato type identity:

Lemma 2.4.1. *For $f(t, x) = t^2 - x_1^2 - \cdots - x_{n-1}^2 \in \mathbb{C}[V]^W$,*

$$(\partial_t^2 - \Delta_k)f^{s+1} = b_k(s)f^s \quad (2.4.1)$$

where

$$b_k(s) = 4(s+1)\left(s + \frac{n}{2} + \gamma_k\right) \quad (2.4.2)$$

Proof. For $s \in \mathbb{N}$, one may make use of the commutation relation $[e_+, e_-] = h$, but a direct calculation yields³

$$\begin{aligned} D_i^2 f^{s+1} &= \left(\partial_i + \sum_{\alpha} \frac{k_{\alpha} \langle \alpha, \xi_i \rangle}{\alpha} (1 - s_{\alpha}) \right) \partial_i f^{s+1} \\ &= \left(\partial_i + \sum_{\alpha} \frac{k_{\alpha} \langle \alpha, \xi_i \rangle}{\alpha} (1 - s_{\alpha}) \right) (s+1)(-2x_i) f^s \\ &= (s+1) \left(\frac{s(-2x_i)^2}{f} + (-2) + \sum_{\alpha} \frac{k_{\alpha} \langle \alpha, \xi_i \rangle}{\alpha} (-2)(x_i - s_{\alpha}(x_i)) \right) f^s. \end{aligned}$$

Recall that $s_{\alpha}(x_i) = x_i - \langle x_i, \alpha^{\vee} \rangle \alpha$, and when summing over i , we see that $\sum_i \langle \alpha, \xi_i \rangle \langle x_i, \alpha^{\vee} \rangle = \langle \alpha, \alpha^{\vee} \rangle = 2$, so

$$\sum_{i=1}^{n-1} D_i^2 f^{s+1} = (s+1) \left(\frac{s \sum 4x_i^2}{f} - 2(n-1) - 4\gamma_k \right) f^s$$

Combined with

$$\partial_t^2 f^{s+1} = (s+1) \left(\frac{s 4t^2}{f} + 2 \right) f^s$$

we conclude (2.4.1) with (2.4.2). □

³The exponent $s \in \mathbb{C}$ is not to be confused with reflections s_{α} .

After all these preliminaries, we shall, assuming $\operatorname{Re} k_\alpha > -\frac{1}{2}$ (and fixed), define the Riesz distribution $R_k^s \in \mathcal{S}'(\mathbb{R}^n)$ for each $s \in \mathbb{C}$ such that $\operatorname{Re} s > \operatorname{Re} s_0$, $s_0 := \frac{n}{2} + \gamma_k - 1$, by

$$R_k^s(t, x) := \frac{1}{\Gamma_{n,k}(s)} \mathcal{T}_k \left((t^2 - |x|^2)^{s - \frac{n}{2} - \gamma_k} \right) \quad (2.4.3)$$

$$= \frac{1}{\Gamma_{n,k}(s)} (t^2 - |x|^2)^{s - \frac{n}{2} - \gamma_k} v_k(x) \quad (t, x) \in C_+ \quad (2.4.4)$$

(and vanishes outside C_+), with

$$\Gamma_{n,k}(s) := c_k 4^s \Gamma(s) \Gamma\left(s - \frac{n}{2} - \gamma_k + 1\right)$$

and

$$v_k(x) = \prod_{\alpha \in \mathcal{R}_+} |\alpha(x)|^{2k_\alpha}$$

Note that, by (2.3.2), we have

$$\begin{aligned} h \cdot R_k^s &= \left(2 \left(s - \frac{n}{2} - \gamma_k \right) + \frac{n}{2} + \gamma_k \right) R_k^s \\ &= \left(2s - \frac{n}{2} - \gamma_k \right) R_k^s. \end{aligned}$$

Recall also that

$$h \cdot \delta = \left(-\frac{n}{2} - \gamma_k \right) \delta$$

which may be taken as an important clue that the exponent in the definition (2.4.3) is the correct one to make $R_k^0 = \delta$.

Theorem 2.4.2. *The family of distributions $R_k^s \in \mathcal{S}'(\mathbb{R}^n)$ is analytic in s (i.e., for each $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\langle R_k^s, \phi \rangle$ is analytic as a function of s) on the region $\operatorname{Re} s > \operatorname{Re} s_0$, and it extends analytically to all $s \in \mathbb{C}$ satisfying*

$$1. (\partial_t^2 - \Delta_k) R_k^{s+1} = R_k^s$$

$$2. R_k^0 = \delta$$

3. $\text{supp } R_k^s = \partial C_+$ for $s \in (s_0 + \mathbb{Z}_{\leq 0}) \setminus \mathbb{Z}_{\leq 0}$

4. $R_k^s *_k R_k^t = R_k^{s+t}$

Remark 2.4.1. It is also interesting to analytically continue R_k^s in k from the region $\text{Re } k_\alpha > -\frac{1}{2}$. It may have poles at various so-called singular values of k that relate to many problems such as finite-dimensional representations of $H_k(W)$, semisimplicity of the Hecke algebra $\mathcal{H}_q(W)$ for $q = e^{2\pi i k}$, and the obstruction to the existence of Dunkl kernel (and Dunkl transform).

The support structure of R_k^s , $s \in \mathbb{C}$, depends on $s_0 = \frac{n}{2} + \gamma_k - 1 \in \mathbb{N}$ or not.

More precisely,

$$\text{supp } R_k^s = \begin{cases} \{0\} & s \in \mathbb{Z}_{\leq 0} \\ \partial C_+ & s \in (s_0 + \mathbb{Z}_{\leq 0}) \setminus \mathbb{Z}_{\leq 0} \\ \bar{C}_+ & \text{otherwise} \end{cases}$$

The following corollary is then immediate:

Corollary 2.4.3. R_k^m is the fundamental solution to the operator $(\partial_t^2 - \Delta_k)^m$, and the Huygens Principle (i.e., $\text{supp } R_k^m \subseteq \partial C_+$) holds if and only if

$$n + 2\gamma_k \text{ is even and } \geq 2 + 2m$$

Remark 2.4.2. Note that the \mathfrak{sl}_2 triple $\{e_+, e_-, h\}$ act on (the \mathbb{C} -span of) the Riesz distributions, and the Huygens Principle holds if R_k^s , $s \in \mathbb{Z}$, exhibit a special structure as an $\mathfrak{sl}_2(\mathbb{C})$ -module, namely there is a short exact sequence

$$0 \rightarrow M_{-\lambda-2} \rightarrow M_\lambda \rightarrow L_\lambda \rightarrow 0$$

where $\lambda = 2s_0 = n + 2\gamma_k - 2$, and $M_{-\lambda-2}$ and M_λ are Verma modules generated by R_k^0 and $R_k^{s_0}$, respectively. L_λ is the finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$ -module of dimension $\lambda + 1$.

Remark 2.4.3. It is worthwhile to make a comparison with Berest-Veselov, which is concerned with

$$\text{Res}(\partial_t^2 - \Delta_k) = \partial_t^2 - \Theta^{-1} L_{k-1} \Theta$$

[BV94] shows that the Huygens Principle holds if and only if $n \geq 4$ is even, $k_\alpha \in \mathbb{Z}_{\geq 0}$, and

$$4 + 2\gamma_k \leq n$$

From [Be00], we have the result for the m -th power:

$$\text{Res}(\partial_t^2 - \Delta_k)^m = \Theta^{-1}(\partial_t^2 - L_{k-1})^m \Theta$$

and the Huygens Principle holds $n \geq 4$ is even, $k_\alpha \in \mathbb{Z}_{\geq 0}$, and

$$\gamma_k < \frac{n}{2m} - 1$$

To compare,

differential case [BV94] [Be00]	Dunkl case [BØ05]
$\begin{cases} n \text{ is even} \\ \text{each } k_\alpha \text{ integral} \\ \frac{n}{2} \geq 1 + m + m\gamma_k \end{cases}$	$\begin{cases} n + 2\gamma_k \text{ is even} \\ \frac{n}{2} + \gamma_k \geq 1 + m \end{cases}$

It is most striking, given Hadamard's condition that n must be even (for a differential operator), is the possibility that n can be odd for Dunkl wave operator. Secondly, for each fixed n and m , the Dunkl case admits more values of k (for various W). Again it seems that $n + 2\gamma_k$ should be regarded as a pseudo-dimension when thinking about Huygens Principle and Hadamard's criterion.

Proof of Theorem. To check (1) over the region $\text{Re } s \gg 0$, we use Bernstein-Sato

identity (2.4.1) and (2.4.2)

$$\begin{aligned}
(\partial_t^2 - \Delta_k)R_k^s &= \frac{1}{\Gamma_{n,k}(s)} \mathcal{T}_k((\partial_t^2 - \Delta_k)f^{s-\frac{n}{2}-\gamma_k}) \\
&= \frac{1}{\Gamma_{n,k}(s)} \mathcal{T}_k\left(b_k\left(s - \frac{n}{2} - \gamma_k - 1\right)f^{s-\frac{n}{2}-\gamma_k-1}\right) \\
&= \frac{1}{\Gamma_{n,k}(s)} 4\left(s - \frac{n}{2} - \gamma_k\right)(s-1) \mathcal{T}_k\left(f^{(s-1)-\frac{n}{2}-\gamma_k}\right) \\
&= \frac{1}{\Gamma_{n,k}(s-1)} \mathcal{T}_k\left(f^{(s-1)-\frac{n}{2}-\gamma_k}\right) \\
&= R_k^{s-1}
\end{aligned}$$

The identity (1) is then used to perform the desired analytic continuation to all $s \in \mathbb{C}$, to wit,

$$R_k^s := (\partial_t^2 - \Delta_k)^m R_k^{s+m}$$

for $\operatorname{Re} s > -m + \operatorname{Re} s_0$.

For (2), we follow the arguments (for $k = 0$) that appear in [KV91] and [DK10]. Consider the family of (Riemann-Liouville) distributions $\chi_+^s \in \mathcal{S}'(\mathbb{R}^1)$, given by the regular function

$$\chi_+^s(x) = \begin{cases} \frac{x^{s-1}}{\Gamma(s)} & x > 0 \\ 0 & x < 0 \end{cases}$$

for $\operatorname{Re} s > 0$, and analytically continued to all $s \in \mathbb{C}$ by means of the identity

$$\frac{d}{dx} \chi_+^s(x) = \chi_+^{s-1}(x)$$

It follows that

$$\chi_+^0 = \frac{d}{dx} \chi_+^1 = \delta$$

and more generally

$$\chi_+^{-j} = \delta^{(j)} \quad j = 0, 1, 2, \dots$$

all supported at the origin.

Now we can identify R_k^s as the pullback of χ_+^s via $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$. More precisely,

$$R_k^s = \frac{1}{c_k 4^s \Gamma(s)} \left(f^* \chi_+^{s - \frac{n}{2} - \gamma_k + 1} \right) v_k(x) \quad \text{on } t > 0$$

because f is a submersion on $\mathbb{R}^n \setminus \{0\}$, and we need to avoid the backward cone. As $\Gamma(s)$ has a (simple) pole at $s = 0$, we see that R_k^0 vanishes as a distribution on the half-space $t > 0$, so it must be supported at $\{0\}$. By a well-known result of Schwartz, R_k^0 must be a finite linear combination of derivatives of the δ -function. On the other hand, the identity

$$h \cdot R_k^s = \left(2s - \frac{n}{2} - \gamma_k \right) R_k^s$$

is valid for $\operatorname{Re} s > \operatorname{Re} s_0$, and as both sides are analytic in s , it must hold for all $s \in \mathbb{C}$. In particular,

$$h \cdot R_k^0 = \left(-\frac{n}{2} - \gamma_k \right) R_k^0$$

Recall that

$$h \cdot \delta = \left(-\frac{n}{2} - \gamma_k \right) \delta$$

but

$$h \cdot \partial_i \delta = \left(-\frac{n}{2} - \gamma_k - 1 \right) \partial_i \delta$$

for any i (and similarly for higher derivatives), so R_k^0 must be a multiple of δ . With the right choice of c_k in the Γ -factor, we can normalize the R_k^s so that $R_k^0 = \delta$. (The precise value of c_k is not important for our purposes; it is closely related to the Mehta constant.) \square

Remark 2.4.4. The classical Riesz distributions R_0^s also satisfy a convolution identity

$$R_0^s * R_0^t = R_0^{s+t} \quad s, t \in \mathbb{C}$$

which does generalize to all (regular) k , where $*$ is replaced by the Dunkl convolution $*_k$. Indeed, the Dunkl analogue of the Fourier transform \mathcal{F}_k converts $*_k$ into

multiplication. Instead of Riesz's definition by means of analytic continuation, the Riesz distributions may also be defined by

$$R_k^s = \mathcal{F}_k \mathcal{T}_k \left(\frac{1}{((t - i0)^2 - |x|^2)^s} \right)$$

It then follows that

$$R_k^s *_k R_k^t = R_k^{s+t} \quad s, t \in \mathbb{C}$$

and one may interpret R_k^s as the complex powers of the Dunkl wave operator.

Having constructed the fundamental solution, we may form what is sometimes called the (Feynman) propagator (for $m = 1$):

$$P_k(t, x) := R_k^1(t, x) - R_k^1(-t, -x)$$

which has the properties that

$$(\partial_t^2 - \Delta_k)P_k(t, x) = \delta - \delta = 0$$

and $\text{supp } P_k = \{|x|^2 = t^2\}$, the full characteristic cone, if $n + 2\gamma_k$ is even and ≥ 4 .

In fact, the proof above gives the explicit expression

$$P_k(t, x) = \frac{1}{4c_k} \delta^{(\ell)}(t^2 - |x|^2) v_k(x), \quad \ell = \frac{n}{2} + \gamma_k - 2$$

away from a neighborhood of the origin. This is the explicit expression that is missing in [BØ05].

2.5 Dunkl analysis by means of hyperfunctions

The standard approach (after Dunkl, de Jeu, Opdam, Trimèche, Rösler, etc.) starts with Dunkl's intertwining operator, then the Dunkl kernel and Dunkl transform,

and finally Dunkl translation and Dunkl convolutions. I will instead start with the Dunkl translation, defined simply as the exponentiation of the Dunkl operator D_ξ . We shall keep in mind a fixed parameter k , which sometimes is hidden from the notations.

Define *Dunkl translation* (on polynomials) by

$$\tau_\eta f = \exp(D_\eta) f \quad \eta \in V \quad f \in \mathbb{C}[V]$$

Proposition 2.5.1. *The Dunkl translation has the following basic properties:*

- 1) D_ξ commutes with τ_η for all $\xi, \eta \in V$.
- 2) $\eta \mapsto \tau_\eta$ defines a group homomorphism $V \rightarrow GL(\mathbb{C}[V])$

This Dunkl translation is used in the definition of Dunkl convolution:

$$(f *_k g)(\xi) = \int_{\eta \in V_{\mathbb{R}}} f(\eta) g(\xi -_k \eta) v_k(\eta) d\eta$$

where $g(\xi -_k \eta) = \tau_{-\eta} g(\xi)$. This obviously doesn't converge if f and g are polynomials, but if properly defined the basic properties such as

$$D_\xi(f *_k g) = (D_\xi f) *_k g = f *_k (D_\xi g)$$

and

$$f *_k g = g *_k f$$

would become tautologies.

Instead of first extending τ_ξ to all C^∞ functions, we shall directly deal with distributions, interpreted as Sato's hyperfunctions. As the theory of hyperfunctions is less well-known (outside of Japan), we shall give an introduction, emphasizing examples over generalities. Some readable references include [KS99], [Sc84] and [Gr10]. The basic idea is to regard distributions on \mathbb{R}^1 such as the δ -function

as the jump of complex-analytic functions on the upper and lower half planes; suggestively,

$$f(x) = F(x + i0) - F(x - i0)$$

and many operations such as differentiation and composition become very natural. The motivating examples are

$$\begin{aligned} \theta(x) &\longleftrightarrow -\frac{1}{2\pi i} \operatorname{Log}(-z) \\ \delta(x) &\longleftrightarrow -\frac{1}{2\pi i} \frac{1}{z} \end{aligned}$$

where $\operatorname{Log} z$ takes the branch cut along the negative real axis with $-\pi < \operatorname{Im} \operatorname{Log}(z) < \pi$. Similarly, with $z^\lambda := e^{\lambda \operatorname{Log} z}$,

$$\chi_+^\lambda(x) = \begin{cases} x^\lambda & x > 0 \\ 0 & x < 0 \end{cases} \longleftrightarrow -\frac{1}{2i \sin(\lambda\pi)} (-z)^\lambda$$

for $\lambda \in \mathbb{C} \setminus \mathbb{Z}$.

More formally, a *hyperfunction* $f(x)$ on \mathbb{R}^1 is given by a complex-analytic function $F(z) \in \mathcal{O}(U \setminus \mathbb{R}^1)$, where $U \subset \mathbb{C}^1$ is an open neighborhood of \mathbb{R}^1 , modulo $\mathcal{O}(U)$. The set of hyperfunctions on \mathbb{R}^1 is denoted

$$\mathcal{B}(\mathbb{R}^1) := \mathcal{O}(U \setminus \mathbb{R}^1) / \mathcal{O}(U)$$

which does not depend on the choice of U . The support of $f(x)$ is defined to be the smallest closed $V \subset \mathbb{R}^1$ such that $F(z)$ admits an analytic continuation to $U \setminus V$.

For a continuous function $f(x)$ of compact support, or more generally $f(x) \in \mathcal{E}'(\mathbb{R}^1)$, it's easy to realize it as a hyperfunction by constructing

$$F(z) = -\frac{1}{2\pi i} \int_{\mathbb{R}} f(x) \frac{1}{z - x} dx$$

outside the support of $f(x)$. Intuitively speaking, all the information of $f(x)$ is encoded in any arbitrarily small (complex) neighborhood, say of 0, in the analytic function $F(z)$.

Now, we can directly apply $\tau_\xi = \exp D_\xi$ to $F(z)$, regarded as a convergent Taylor or Laurent series in the (upper and lower) neighborhood of 0. For example, with $D_\zeta = \zeta(\frac{\partial}{\partial z} + \frac{k}{z}(1-s))$,

$$\tau_\zeta \frac{1}{z} = \frac{1}{z} + (-1-2k) \frac{\zeta}{z^2} + \frac{(-1-2k)(-2)}{2!} \frac{\zeta^2}{z^3} + \dots$$

which for $k=0$ converges to simply $\frac{1}{z+\zeta}$ for $|\zeta| < |z|$, which for real ζ would be a delta function at $-\zeta$.

It is more difficult to formally define hyperfunctions on \mathbb{R}^n for $n > 1$, which heavily relies on sheaf cohomology, but the idea of boundary values still works. To illustrate it with $n=2$, we may write

$$f(x_1, x_2) = F(x_1+i0, x_2+i0) - F(x_1-i0, x_2+i0) - F(x_1+i0, x_2-i0) + F(x_1-i0, x_2-i0)$$

where $F(z_1, z_2) = \frac{1}{z_1 z_2}$, for instance.

2.6 Interlude: other constructions of P_k

In passing, we briefly mention other constructions of the wave equation, adapted to the Dunkl operators. One elegant method, used by [BØ05], is to write the wave equation for u as a first-order system of equations for u and an auxiliary unknown function v :

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta_k & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Apply the Dunkl transform (in the x variables):

$$u(t, x) \mapsto \hat{u}(t, \xi) = (\mathcal{F}_k u)(t, \xi) := \int_{\mathbb{R}^{n-1}} E_k(x, -i\xi) u(t, x) v_k(x) dx$$

we get

$$\partial_t \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -|\xi|^2 & 0 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$$

which is solved (as an ODE, with $\xi \in \mathbb{R}^{n-1}$ fixed) by

$$\begin{aligned} \begin{pmatrix} \hat{u}(t) \\ \hat{v}(t) \end{pmatrix} &= \exp \left\{ t \begin{pmatrix} 0 & 1 \\ -|\xi|^2 & 0 \end{pmatrix} \right\} \begin{pmatrix} \hat{u}_0 \\ \hat{v}_0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(t|\xi|) & \frac{\sin(t|\xi|)}{|\xi|} \\ -|\xi| \sin(t|\xi|) & \cos(t|\xi|) \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{v}_0 \end{pmatrix} \end{aligned}$$

Applying inverse Dunkl transform, we get

$$u(t) = \mathcal{F}_k^{-1}[\cos(t|\xi|)] *_k u_0 + \mathcal{F}_k^{-1}\left[\frac{\sin(t|\xi|)}{|\xi|}\right] *_k v_0$$

To justify it, we can show that the two distributions in the expression are supported in the ball of radius $|t|$ in \mathbb{R}^{n-1} by the Dunkl version of Paley-Wiener theorem, so the convolution $*_k$ makes sense. In other words, as distributions in $\mathcal{S}'(\mathbb{R}^n)$, the supports are contained in the (double) cone $\{(t, x) : |x|^2 \leq t^2\}$. Notice that the first distribution may be obtained from the second one by differentiating in t , so we may write

$$u(t) = \partial_t P_k(t) *_k u_0 + P_k(t) *_k v_0$$

where

$$P_k(t) = \mathcal{F}_k^{-1}\left[\frac{\sin(t|\xi|)}{|\xi|}\right](x)$$

is precisely the propagator P_k from §2.4.

Alternatively, we may construct the fundamental solution via the general procedure of Fourier-Laplace transform in (t, x) . Adapted to Dunkl operators, we

have

$$E_{\pm} = \frac{1}{C_{n,k}} \int_{\mathbb{R}^n} \frac{e^{it(\tau \pm i\varepsilon)} E_k(x, i\xi)}{-(\tau \pm i\varepsilon)^2 + |\xi|^2} v_k(\xi) d\tau d\xi \quad \varepsilon > 0$$

By combining the two integrals, and deforming the chain of integration, we are able to perform the (complex) τ integral by computing the residues at $\tau = \pm|\xi|$. The result agrees:

$$E_+ - E_- = \frac{1}{C_{n-1,k}} \int_{\mathbb{R}^{n-1}} \frac{\sin(t|\xi|)}{|\xi|} E_k(x, i\xi) v_k(\xi) d\xi = \mathcal{F}_k^{-1} \left[\frac{\sin(t|\xi|)}{|\xi|} \right]$$

2.7 The Cauchy problem

We consider the wave operator with Calogero-Moser potential

$$\partial_t^2 - L_k = \partial_t^2 - \Delta + \sum_{\alpha \in \mathcal{R}_+} \frac{k_{\alpha}(k_{\alpha} + 1)(\alpha, \alpha)}{(\alpha, x)^2}$$

where the potential is singular along each of the reflecting hyperplanes. It is natural then to consider the Cauchy problem over a Weyl chamber, i.e., on $\mathbb{R} \times \Omega$, where

$$\Omega := \{x \in \mathbb{R}^{n-1} : \alpha(x) > 0 \text{ for all } \alpha \in \mathcal{R}_+\}.$$

(We only need $\alpha \in \Pi$, the simple roots for the given positive system \mathcal{R}_+ .)

Theorem 2.7.1. *The Cauchy-Dirichlet problem*

$$\begin{cases} (\partial_t^2 - L_k)u = 0 & \text{on } \mathbb{R} \times \Omega \\ u(0, x) = f_0(x) & \text{on } \Omega \\ \partial_t u(0, x) = f_1(x) & \text{on } \Omega \\ u(t, x) = 0 & \text{on } \mathbb{R} \times \partial\Omega \end{cases} \quad (2.7.1)$$

with initial data $f_0, f_1 \in \mathcal{D}'(\Omega)$, has a unique solution $u \in \mathcal{D}'(\mathbb{R} \times \Omega)$ (global well-posedness after Hadamard). Moreover, the solution $u(t, x)$ satisfies a weak Huygens

Principle in the sense that $u(t, x)$ vanishes on the domain

$$\{(t, x) \in \mathbb{R} \times \Omega : |t| > \sup_{w \in W, y \in S} |x - wy|\}$$

for all f_0 and f_1 supported in a fixed bounded set $S \subset \Omega$, provided that

$$n + 2\gamma_k = 4, 6, \dots$$

Proof. Along the same line as in §2.1, we extended the (modified) initial data $\tilde{f}_i := \Theta^{-1} f_i$, where

$$\Theta = \Theta_k(x) := \prod_{\alpha \in \mathcal{A}_+} \alpha(x)^{2k_\alpha} \quad x \in \Omega$$

from Ω to the whole \mathbb{R}^{n-1} as W -alternating functions (or distributions), and solve the associated Cauchy problem of Dunkl wave equation

$$\begin{cases} (\partial_t^2 - \Delta_k) \tilde{u} = 0 & \text{on } \mathbb{R}^n \\ \tilde{u}(0, x) = \tilde{f}_0(x) & \text{on } \mathbb{R}^{n-1} \\ \partial_t \tilde{u}(0, x) = \tilde{f}_1(x) & \text{on } \mathbb{R}^{n-1} \end{cases} \quad (2.7.2)$$

which is solved by Dunkl convolution:

$$\tilde{u}(t) := \partial_t P_k(t) *_k \tilde{f}_0 + P_k(t) *_k \tilde{f}_1$$

where $P_k(t)$ denotes the t -section of the distribution

$$P_k(t, x) := R_k^1(t, x) - R_k^1(-t, -x)$$

We only need to show that

$$u(t, x) := \Theta(x) \tilde{u}(t, x) \quad x \in \Omega$$

solves (2.7.1). First we claim that \tilde{u} is W -alternating. For convenience, we shall assume $\tilde{f}_0 \equiv 0$ and only worry about the term involving \tilde{f}_1 . At least formally, we

have

$$\begin{aligned}
s_\alpha \tilde{u}(t, x) &= \int P_k(t, s_\alpha x -_k y) \tilde{f}_1(y) dy \\
&= \int P_k(t, s_\alpha x -_k s_\alpha y) \tilde{f}_1(s_\alpha y) d(s_\alpha y) \\
&= \int P_k(t, x -_k y) (-\tilde{f}_1(y)) dy = -\tilde{u}(t, x)
\end{aligned}$$

because P_k is W -invariant and \tilde{f}_1 is W -alternating.

We then have, by Lemma 2.2.2 with ε the sign character,

$$\Theta^{-1}(\partial_t^2 - L_k)\Theta \tilde{u} = (\partial_t^2 - \Delta_k)\tilde{u} = 0$$

which shows that $u = \Theta \tilde{u}$ solves (2.7.1).

To show the weak Huygens Principle for $n + 2\gamma_k = 4, 6, 8, \dots$, we only need to show that

$$P_k(t) *_k \tilde{f}_1$$

has the desired support structure if the distribution P_k is supported in $C_+ \cup C_-$.

This is given in [BØ05] by analytical means.

The uniqueness of the solution is also given in [BØ05] by using a suitable energy functional. □

A similar result works for Neumann boundary condition, i.e.

$$\partial_\alpha u(t, x) = 0 \quad (t, x) \in \mathbb{R} \times \partial\Omega \quad \text{such that} \quad \alpha(x) = 0$$

for each simple root $\alpha \in \Pi$, in place of Dirichlet condition. We simply need to replace the sign character by the trivial character, and reflect the initial conditions f and g as W -invariant functions. Slightly more generally (for W with more than one conjugacy class of reflections), given any character $\varepsilon : W \rightarrow \{\pm 1\}$, we can have

either Dirichlet or Neumann boundary condition for a “wall” $\{x \in \partial\Omega \mid \alpha(x) = 0\}$, according to whether $\varepsilon(s_\alpha) = -1$ or $\varepsilon(s_\alpha) = 1$, respectively. The solution is simply given by the same expression, with k in the distribution P_k replaced by k^ε (but $\Theta = \Theta_k(x)$ remains the same). The condition for weak Huygens Principle becomes

$$n + 2 \sum_{\alpha \in \mathcal{R}_+} k_\alpha^\varepsilon = n + 2 \sum_{\alpha \in \mathcal{R}_+} \left(k_\alpha - \frac{\varepsilon(s_\alpha) + 1}{2} \right) = 4, 6, \dots$$

CHAPTER 3

A HIGHER ORDER OPERATOR

It is natural to look for operators other than powers of the Dunkl wave operator, that the techniques of Riesz distributions for the Dunkl wave operators could apply.

3.1 Bernstein-Sato type identities

It is clear from the proof in §2.4 that the Bernstein-Sato identity for Dunkl operators is essential to our new Riesz distributions, in the very definition as well as in the support structure. To test the idea, we need other examples of

$$P(\partial)f^{s+1} = b(s)f^s$$

and if one replaces ∂ by Dunkl operators D (for a particular W and arbitrary k), can we expect a similar identity

$$P(D)f^{s+1} = b_k(s)f^s$$

where

$$b_k(s) \in \mathbb{C}[s]$$

that depends on k ? It is necessary that f be W -invariant, so P also needs to be W -invariant.

It is well-known (Chevalley's Theorem) that the algebra $\mathbb{C}[V]^W$ of W -invariant polynomial is free, so it makes sense to first look for the generators of $\mathbb{C}[V]^W$. For $W = S_n$ acting on $V \cong \mathbb{C}^n$, one convenient set of generators are the elementary symmetric polynomials $\sigma_r(x)$, $r = 1, \dots, n$:

$$\sigma_r(x) = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Unfortunately, the operator P that is guaranteed by Bernstein's theorem is rarely constant coefficient.

For $r = 1$,

$$\sigma_1(x) = x_1 + \cdots + x_n$$

which does satisfy

$$(D_1 + \cdots + D_n)\sigma_1^{s+1} = n\sigma_1^s$$

(i.e. $P = \sigma_1$).

For $r = 2$, one can check directly that, with

$$P = \sigma_2 - \frac{n-2}{2(n-1)}\sigma_1^2$$

as the operator,

$$P(D)\sigma_2^{s+1} = (s+1)\left(s + \frac{n}{2} + \frac{n(n-1)}{2}k\right)\sigma_2^s$$

with D the Dunkl operators for $W = S_n$ and parameter $k \in \mathbb{C}$. In fact, $P(D)$ is precisely the Dunkl wave equation in dimension n , with $(1, \dots, 1)$ the time direction, and the identity is identical to that in Lemma 2.4.2 because

$$\gamma_k = \frac{n(n-1)}{2}k$$

for type A .

From $r = 3$ on, such constant coefficient operators P do not exist, except for σ_n , the top-degree elementary symmetric function, which is the product of the coordinates:

$$\sigma_n(x) = x_1 \cdots x_n$$

Proposition 3.1.1 (Type A_{n-1}). *With $f = \sigma_n$, the following identity holds*

$$D_1 \cdots D_n f^{s+1} = \left(\prod_{j=0}^{n-1} (s+1+jk) \right) f^s$$

Remark 3.1.1. As with Lemma 2.4.2, this is an algebraic identity inside the $H_k(W)$ -module

$$M_f = \bigoplus_{s \in \mathbb{C}} \mathbb{C}[V] f^s / \sim$$

generated by the formal symbols f^s , modulo the relations $f f^s = f^{s+1}$. The action of $H_k(W)$ is defined by

$$D_i \cdot (p f^s) = (D_i p) f^s + s(p D_i f) f^{s-1}$$

(note that $D_i f = \partial_i f$ since f is W -invariant) and obvious actions of x_i and s_α . When doing calculations, we may pretend f to be an actual function on an open subset of $V_{\mathbb{R}}$ over which f is positive, so f^s is literally the complex power with the usual branch cut.

Note that M_f and M_{f^2} are *not* considered the same $H_k(W)$ -module.

Proof. Since the D_i 's commute, we may perform the calculation in the reverse order, i.e., with $P(D) = D_n \cdots D_2 D_1$. In fact, we shall prove

$$D_i \cdots D_1 f^{s+1} = \left(\prod_{j=0}^{i-1} (s+1+jk) \right) \frac{f^{s+1}}{x_1 \cdots x_i} \quad (3.1.1)$$

for each $i = 1, \dots, n$, by a simple induction on i (with n fixed). Note that on the right hand is actually a shorthand:

$$\frac{f^{s+1}}{x_1 \cdots x_i} = \frac{f}{x_1 \cdots x_i} f^s = x_{i+1} \cdots x_n f^s \in \mathbb{C}[V] f^s$$

It is clearly true for $i = 1$, and for the inductive step, we shall apply the next Dunkl operator

$$D_{i+1} = \frac{\partial}{\partial x_{i+1}} + \sum_{j \neq i+1} \frac{k}{x_{i+1} - x_j} (1 - s_{i+1,j})$$

to (3.1.1). Note that

$$s_{i+1,j} \frac{1}{x_1 \cdots x_i} = \begin{cases} \frac{x_j}{x_{i+1}} \frac{1}{x_1 \cdots x_i} & 1 \leq j \leq i \\ \frac{1}{x_1 \cdots x_i} & i+2 \leq j \leq n \end{cases}$$

so we have

$$\begin{aligned} D_{i+1} \frac{f^{s+1}}{x_1 \cdots x_i} &= \left(\frac{s+1}{x_{i+1}} + \sum_{1 \leq j \leq i} \frac{k}{x_{i+1} - x_j} \left(1 - \frac{x_j}{x_{i+1}}\right) \right) \frac{f^{s+1}}{x_1 \cdots x_i} \\ &= (s+1 + ik) \frac{f^{s+1}}{x_1 \cdots x_i x_{i+1}} \end{aligned}$$

as desired. Putting $i = n$ in (3.1.1) yields the lemma. \square

For give an idea of the complication behind the simple appearance $P(D) = D_1 \cdots D_n$, we compute the restriction of $D_1 D_2 D_3$ on S_3 -invariant functions, and the result is

$$\begin{aligned} &\partial_1 \partial_2 \partial_3 + \frac{k}{x_1 - x_2} (\partial_2 - \partial_1) \partial_3 + \frac{k}{x_2 - x_3} (\partial_3 - \partial_2) \partial_1 + \frac{k}{x_3 - x_1} (\partial_1 - \partial_3) \partial_2 \\ &+ \frac{k^2}{(x_1 - x_2)(x_1 - x_3)} (\partial_1 - \partial_2) + \frac{k^2}{(x_2 - x_3)(x_2 - x_1)} (\partial_2 - \partial_3) \\ &+ \frac{k^2}{(x_3 - x_1)(x_3 - x_2)} (\partial_3 - \partial_1) \end{aligned}$$

If we put $f(x) = \sigma_n(x)^2 = x_1^2 \cdots x_n^2$, it is also invariant under $W = S_n \ltimes (\mathbb{Z}_2)^n$, the Coxeter group of type B_n . Recall that the Dunkl operators for type B take the form

$$D_i = \frac{\partial}{\partial x_i} + \frac{k}{x_i} (1 - s_i) + \sum_{j \neq i} \left(\frac{\kappa}{x_i - x_j} (1 - s_{ij}) + \frac{\kappa}{x_i + x_j} (1 - r_{ij}) \right)$$

where the reflections act by $s_i(x_i) = -x_i$, $s_{ij}(x_i) = x_j$, $s_{ij}(x_j) = x_i$, $r_{ij}(x_i) = -x_j$, $r_{ij}(x_j) = -x_i$ and leaving all other coordinates fixed. As before, to better book-keep the calculations we rewrite differentiation and reflections as multiplication by rational functions.

Proposition 3.1.2 (Type B_n). For $f(x) = x_1^2 \cdots x_n^2$, we have

$$D_1^2 \cdots D_n^2 f^{s+1} = 4^n \left(\prod_{j=0}^{n-1} (s+1+j\kappa)(s+\frac{1}{2}+k+j\kappa) \right) f^s$$

for any parameters $k, \kappa \in \mathbb{C}$.

Proof. Similarly we shall prove

$$D_i^2 \cdots D_1^2 f^{s+1} = 4^i \left(\prod_{j=0}^{i-1} (s+1+j\kappa)(s+\frac{1}{2}+k+j\kappa) \right) \frac{f^{s+1}}{x_1^2 \cdots x_i^2}$$

for each $i = 1, \dots, n$. For the base case $i = 1$, we compute

$$\begin{aligned} D_1 f^{s+1} &= 2(s+1) \frac{f^{s+1}}{x_1} \\ D_1^2 f^{s+1} &= 2(s+1) \left(\frac{2(s+1)}{x_1} - \frac{1}{x_1} + \frac{2k}{x_1} \right. \\ &\quad \left. + \sum_{j \neq 1} \frac{\kappa}{x_1 - x_j} (1 - \frac{x_1}{x_j}) + \frac{\kappa}{x_1 + x_j} (1 - \frac{x_1}{-x_j}) \right) \frac{f^{s+1}}{x_1} \\ &= 2(s+1) 2(s+\frac{1}{2}+k) \frac{f^{s+1}}{x_1^2} \end{aligned}$$

and for the inductive step, we need to compute

$$\begin{aligned} D_{i+1} \frac{f^{s+1}}{x_1^2 \cdots x_i^2} &= \left(\frac{2(s+1)}{x_{i+1}} + \sum_{j < i+1} \left(\frac{\kappa}{x_{i+1} - x_j} + \frac{\kappa}{x_{i+1} + x_j} \right) (1 - \frac{x_j^2}{x_{i+1}^2}) \right) \frac{f^{s+1}}{x_1^2 \cdots x_i^2} \\ &= 2(s+1+i\kappa) \frac{f^{s+1}}{x_1^2 \cdots x_i^2 x_{i+1}} \\ D_{i+1}^2 \frac{f^{s+1}}{x_1^2 \cdots x_i^2} &= 2(s+1+i\kappa) \left(\frac{2(s+1)}{x_{i+1}} - \frac{1}{x_{i+1}} + \frac{2k}{x_{i+1}} \right. \\ &\quad \left. + \sum_{j < i+1} \frac{\kappa}{x_{i+1} - x_j} (1 - \frac{x_j}{x_{i+1}}) + \frac{\kappa}{x_{i+1} + x_j} (1 + \frac{x_j}{x_{i+1}}) \right. \\ &\quad \left. + \sum_{j > i+1} \frac{\kappa}{x_{i+1} - x_j} (1 - \frac{x_{i+1}}{x_j}) + \frac{\kappa}{x_{i+1} + x_j} (1 + \frac{x_{i+1}}{x_j}) \right) \frac{f^{s+1}}{x_1^2 \cdots x_i^2 x_{i+1}} \\ &= 2(s+1+i\kappa) 2(s+\frac{1}{2}+k+i\kappa) \frac{f^{s+1}}{x_1^2 \cdots x_{i+1}^2} \end{aligned}$$

□

We may go further by considering the complex reflection group $G(m, 1, n) = S_n \ltimes (\mathbb{Z}_m)^n$, which for $m = 1$ and $m = 2$ gives Coxeter groups of type A_{n-1} and B_n , respectively. See §3.3 for the definitions and the Bernstein-Sato type identity for the general case $G(m, p, n)$.

3.2 Riesz distributions for $P(D) = D_1 \cdots D_n$

Switching back to $W = S_n$ of type A , we can construct the Riesz distributions for $P(D) = D_1 D_2 \cdots D_n$ to obtain distributions supported on each of the singular strata

$$\{0\} = X_0 \subset X_1 \subset \cdots \subset X_n = \bar{C}_+$$

$$X_j := \bigcup_{I \subseteq \{1, \dots, n\}, |I|=j} \{x \in \bar{C}_+ \mid x_i = 0 \text{ for } i \notin I\}$$

of the “forward” propagation cone $\bar{C}_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0 \ \forall i\}$. For $s \in \mathbb{C}$ such that $\operatorname{Re} s > (n-1)\operatorname{Re} k$, define $R_k^s \in \mathcal{S}'(\mathbb{R}^n)$ by the expression

$$R_k^s = \frac{1}{\Gamma_{n,k}(s)} (x_1 \cdots x_n)^{s-(n-1)k-1} v_k(x) \quad x \in C_+$$

(and vanishes outside C_+), where

$$\Gamma_{n,k}(s) = c_k \prod_{j=0}^{n-1} \Gamma(s - jk)$$

and

$$v_k(x) = \prod_{i < j} |x_i - x_j|^{2k}$$

Theorem 3.2.1. *The map $s \mapsto R_k^s \in \mathcal{S}'(\mathbb{R}^n)$ can be analytically continued to all $s \in \mathbb{C}$ such that*

$$1. \ D_1 \cdots D_n R_k^s = R_k^{s-1}$$

$$2. R_k^0 = \delta$$

$$3. \text{supp } R_k^{jk} = X_j \text{ for } j = 0, \dots, n \text{ (except for } k = 0)$$

Remark 3.2.1. The classical case ($k = 0$) was considered by Herriot [Her55], following closely Riesz's original techniques.

Corollary 3.2.2. *The fundamental solution to $D_1 \cdots D_n$ supported in \bar{C}_+ is R_k^1 , and it satisfies the Huygens Principle if and only if $k = \frac{p}{q} \in \mathbb{Q}_{>0}$ with $(p, q) = 1$ and $1 \leq q \leq n - 1$ (more precisely, $\text{supp } R_k^1 = X_q$).*

Proof. Using the Bernstein-Sato type identity for $f = x_1 \cdots x_n$, we have

$$\begin{aligned} D_1 \cdots D_n R_k^s &= \frac{1}{\Gamma_{n,k}(s)} \mathcal{T}_k (D_1 \cdots D_n f^{s-(n-1)k-1}) \\ &= \frac{1}{\Gamma_{n,k}(s)} \mathcal{T}_k (b_k(s - (n-1)k - 2) f^{s-(n-1)k-2}) \\ &= \frac{1}{\Gamma_{n,k}(s)} \prod_{j=0}^{n-1} (s - 1 - jk) \mathcal{T}_k (f^{s-(n-1)k-2}) \\ &= \frac{1}{\Gamma_{n,k}(s-1)} \mathcal{T}_k (f^{(s-1)-(n-1)k-1}) \\ &= R_k^{s-1} \end{aligned}$$

which is used to analytically continue R_k^s to all $s \in \mathbb{C}$. We also note that

$$\begin{aligned} h \cdot R_k^s &= \left(n(s - (n-1)k - 1) + \frac{n}{2} + \gamma_k \right) R_k^s \\ &= \left(ns - \frac{n}{2} - \gamma_k \right) R_k^s \end{aligned}$$

(recall that $\gamma_k = \frac{n(n-1)}{2}k$), which holds for all $s \in \mathbb{C}$ by analytic continuation. If we can show that $\text{supp } R_k^0 = \{0\}$, then by the same argument as before we conclude that $R_k^0 = \delta$ for the right choice of the constant c_k .

To study the support of R_k^s , we first note that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a submersion outside the singular locus of the hypersurface $f^{-1}(0)$

$$\text{Sing } f^{-1}(0) = \{x \in \mathbb{R}^n \mid \text{at least two of the } x_i \text{'s are zero}\}$$

(which contains X_{n-2} and its various reflections). Along the same line as before, we see that

$$R_k^s = \frac{1}{c \prod_{j=0}^{n-2} \Gamma(s - jk)} f^* \chi_+^{s-(n-1)k} v_k(x) \quad \text{away from } X_{n-2}$$

which implies that

$$\text{supp } R_k^{(n-1)k} = X_{n-1}$$

and that

$$\text{supp } R_k^{jk} \subseteq X_{n-2} \quad j = 0, \dots, n-2$$

To obtain more refined result on the support, we note that the complex power $(x_1 \dots x_n)^s$ naturally splits into product of many powers, each with a subset of the variables. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of n , i.e., $\lambda_i \subset \{1, \dots, n\}$ are disjoint and $\bigcup \lambda_i = \{1, \dots, n\}$. Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by $f_i(x) = \prod_{j \in \lambda_i} x_j$, so that $f = \prod_{i=1}^r f_i$

$$R_k^s = \frac{\Gamma(s - (n-1)k)^{r-1}}{c_k \prod_{j=0}^{n-2} \Gamma(s - jk)} \prod \left(f_i^* \chi_+^{s-(n-1)k}(x) \right) v_k(x)$$

on any domain over which f_i are all submersions.

□

One could also consider $W = S_m$ acting on the first $m < n$ variables, which only splits $m-1$ of the roots of Bernstein-Sato polynomial away from $s = -1$.

3.3 Dunkl operators for complex reflection groups

Dunkl operators in §2.2 can be generalized to complex reflection groups (denoted G instead of W). The definition in Dunkl-Opdam [DO03] is

$$D_\xi := \partial_\xi + \sum_H \sum_{i=0}^{n_H-1} \frac{n_H k_{H,i} \langle \alpha_H, \xi \rangle}{\alpha_H} e_{H,i} \quad \xi \in V$$

where H runs over a set of hyperplanes in V , $\alpha_H \in V^*$ that vanishes on H , and for each $i = 0, 1, \dots, n_H - 1$, $e_{H,i} \in \mathbb{C}[G]$ is the idempotent of the isotropy subgroup $G_H \cong \mathbb{Z}_{n_H} = \mathbb{Z}/n_H\mathbb{Z}$. The parameters satisfy $k_{H,i} = k_{H',i}$ if $H = gH'$ for some $g \in G$, and it's typically assumed that $k_{H,0} = 0$ so that D_ξ act on polynomials.

Alternatively, we may choose another set of parameters c_s , one for each of $s \in S$, the set of complex (or pseudo-) reflections in G , subject to the condition $c_s = c_{s'}$ if s and s' are conjugate in G , and write simply

$$D_\xi := \partial_\xi + \sum_{s \in S} \frac{c_s \langle \alpha_s, \xi \rangle}{\alpha_s} (1 - s) \quad \xi \in V$$

(a correspondence of $\{c_s\}$ with $\{k_{H,i}\}$ can be established). The definition relies on the following lemma.

Lemma 3.3.1. *For a complex reflection group G on V and V^* , there exists a root system $\{\alpha_s\}_{s \in S} \subset V^*$ and a coroot system $\{\alpha_s^\vee\}_{s \in S} \subset V$, such that*

1. $\langle \alpha_s, \alpha_s^\vee \rangle = 1 - \lambda_s$ where λ_s is a primitive m -th root of unity, m being the order of s in G

2.

$$s(x) = x - \langle x, \alpha_s^\vee \rangle \alpha_s \quad s^{-1}(\xi) = \xi - \langle \alpha_s, \xi \rangle \alpha_s^\vee$$

for all $s \in S$, $x \in V^*$ and $\xi \in V$

3. $s(\alpha_t) = \alpha_{sts^{-1}}$ for all $s, t \in S$

Then the following theorem is almost identical to the real reflection groups.

Theorem 3.3.2 (Dunkl-Opdam). *The Dunkl operators for complex reflection groups satisfy the following relations*

1. $D_\xi D_\eta = D_\eta D_\xi$ (commutativity)
2. $sD_\xi = D_{s(\xi)}s$ (G -equivariance)
3. $[D_\xi, x] = \langle x, \xi \rangle + \sum_{s \in S} c_s \langle x, \alpha_s^\vee \rangle \langle \alpha_s, \xi \rangle s$

for all $\xi, \eta \in V$, $x \in V^*$ and $s \in S$.

Proof. For (3),

$$\begin{aligned}
[D_\xi, x] &= [\partial_\xi, x] + \sum_{s \in S} \frac{c_s \langle \alpha_s, \xi \rangle}{\alpha_s} (xs - sx) \\
&= \langle x, \xi \rangle + \sum_{s \in S} \frac{c_s \langle \alpha_s, \xi \rangle}{\alpha_s} (xs - (x - \langle x, \alpha_s^\vee \rangle \alpha_s)s) \\
&= \langle x, \xi \rangle + \sum_{s \in S} c_s \langle x, \alpha_s^\vee \rangle \langle \alpha_s, \xi \rangle s
\end{aligned}$$

For (2),

$$\begin{aligned}
sD_\xi &= \partial_{s(\xi)}s + \sum_{t \in S} \frac{c_t \langle \alpha_t, \xi \rangle}{s(\alpha_t)} s(1 - t) \\
&= \partial_{s(\xi)}s + \sum_{t \in S} \frac{c_t \langle \alpha_t, \xi \rangle}{s(\alpha_t)} (1 - sts^{-1})s \\
&= \left(\partial_{s(\xi)} + \sum_{t \in S} \frac{c_t \langle s^{-1}(\alpha_r), \xi \rangle}{\alpha_r} (1 - r) \right) s
\end{aligned}$$

where $r := sts^{-1}$, and $\alpha_r = s(\alpha_t)$. Then (2) follows from $c_t = c_r$ and $\langle s^{-1}(\alpha_r), \xi \rangle = \langle \alpha_r, s(\xi) \rangle$.

For (1), we note that

$$[[D_\xi, D_\eta], x] = [[D_\xi, x], D_\eta] - [[D_\eta, x], D_\xi]$$

and using (2) and (3),

$$\begin{aligned}
[[D_\xi, x], D_\eta] &= [\langle x, \xi \rangle + \sum_{s \in S} c_s \langle x, \alpha_s^\vee \rangle \langle \alpha_s, \xi \rangle s, D_\eta] \\
&= \sum_{s \in S} c_s \langle x, \alpha_s^\vee \rangle \langle \alpha_s, \xi \rangle [s, D_\eta] \\
&= \sum_{s \in S} c_s \langle x, \alpha_s^\vee \rangle \langle \alpha_s, \xi \rangle s \langle \alpha_s, \eta \rangle D_{\alpha_s^\vee}
\end{aligned}$$

Since ξ and η appear symmetrically it follows that $[[D_\xi, D_\eta], x] = 0$. This means that for any $f \in \mathbb{C}[V]$, $[D_\xi, D_\eta]f = f[D_\xi, D_\eta]1 = 0$, hence $[D_\xi, D_\eta] = 0$ (since $\mathbb{C}[V]$ is a faithful representation). \square

The classification of (irreducible) complex reflection groups was achieved by Shephard and Todd [ST54], who used it to check that $\mathbb{C}[V]^G$ is free case-by-case. The list consists of an infinite family $G(m, p, n)$ of three integer parameters with $p \mid m$, and 34 exceptional groups. For special cases, $G(1, 1, n)$, $G(2, 1, n)$, $G(2, 2, n)$, and $G(m, 1, 2)$ are Coxeter groups of types A_{n-1} , B_n , D_n , and $I_2(m)$, respectively (see §2.2).

The groups $G(m, 1, n)$ are the wreath product $S_n \ltimes (\mathbb{Z}_m)^n$, and one can in fact write the Dunkl operators more explicitly using coordinates [DO03]:

$$D_i = \frac{\partial}{\partial x_i} + \kappa \sum_{j \neq i} \sum_{l=0}^{m-1} \frac{1 - s_i^{-l} s_{ij} s_i^l}{x_i - \eta^l x_j} + \sum_{p=1}^{m-1} k_p \sum_{l=0}^{m-1} \frac{\eta^{-pl} s_i^l}{x_i}$$

where $\eta = e^{2\pi i/m}$ is an m -th root of unity, $s_i(x_i) = \eta x_i$ (and leaving other coordinates fixed) is an order- m reflection. The parameters are $\kappa, k_1, \dots, k_{m-1} \in \mathbb{C}$ (κ is associated with the order 2 reflection s_{ij} , while k_1, \dots, k_{m-1} are for the order m reflection s_i).

In fact, we state the Bernstein-Sato type identity for the general group $G(m, p, n)$ with $p \mid m$. The Dunkl operators take the same form, with the fol-

lowing restriction on the parameters

$$k_{tm/p} = 0 \quad k_r = k_{r+tm/p} \quad t = 1, \dots, p-1$$

Theorem 3.3.3 (general $G(m, p, n)$). *For $f(x) = \sigma_n^{m/p}(x) = x_1^{m/p} \cdots x_n^{m/p}$, we have*

$$D_1^m \cdots D_n^m f^{s+1} = b_k(s) f^s$$

where

$$b_k(s) = \left(\frac{m}{p}\right)^{\frac{mn}{p}} \prod_{j=0}^{n-1} \prod_{r=1}^{m/p} \left(s + \frac{r}{m/p} + j\kappa + k_r\right)$$

(with $k_m = 0$).

Proof. We prove it for $p = 1$, and the general case is a slight modification. With the experience of type A_{n-1} and B_n , it is not too hard to perform the calculation.

We shall need repeatedly the formula

$$\sum_{l=0}^{m-1} \eta^{rl} = \begin{cases} m & r \equiv 0 \pmod{m} \\ 0 & r \not\equiv 0 \pmod{m} \end{cases}$$

$$D_1 f^{s+1} = \frac{m(s+1)}{x_1} f^{s+1}$$

...

$$\begin{aligned} D_1^{r+1} f^{s+1} &= m(s+1) \cdots \left(\frac{m(s+1)}{x_1} - \frac{r}{x_1} + \kappa \sum_{j \neq 1}^{m-1} \sum_{l=0}^{m-1} \frac{1 - \frac{x_1^r}{(\eta^l x_j)^r}}{x_1 - \eta^l x_j} + \right. \\ &\quad \left. \sum_{p=1}^{m-1} k_p \sum_{l=0}^{m-1} \frac{\eta^{-pl} \eta^{-rl}}{x_1} \right) \frac{f^{s+1}}{x_1^r} \\ &= m(s+1) \cdots m\left(s + \frac{m-r}{m} + k_{m-r}\right) \frac{f^{s+1}}{x_1^{r+1}} \end{aligned}$$

...

$$D_1^m f^{s+1} = m^m \left(\prod_{r=1}^m \left(s + \frac{r}{m} + k_r\right) \right) \frac{f^{s+1}}{x_1^m}$$

if we put $k_m = 0$. Continuing,

$$\begin{aligned}
D_{i+1} \frac{f^{s+1}}{x_1^m \cdots x_i^m} &= \left(\frac{m(s+1)}{x_{i+1}} + \kappa \sum_{j < i+1} \sum_{l=0}^{m-1} \frac{1 - \frac{x_j^m}{(\eta^{-l} x_{i+1})^m}}{x_{i+1} - \eta^l x_j} \right) \frac{f^{s+1}}{x_1^m \cdots x_i^m} \\
&= m(s+1 + i\kappa) \frac{f^{s+1}}{x_1^m \cdots x_i^m x_{i+1}} \\
&\dots \\
D_{i+1}^{r+1} \frac{f^{s+1}}{x_1^m \cdots x_i^m} &= m(s+1 + i\kappa) \cdots \left(\frac{m(s+1)}{x_{i+1}} - \frac{r}{x_{i+1}} + \kappa \sum_{j < i+1} \sum_{l=0}^{m-1} \frac{1 - \frac{x_j^{m-r}}{(\eta^{-l} x_{i+1})^{m-r}}}{x_{i+1} - \eta^l x_j} \right. \\
&\quad \left. + \kappa \sum_{j > i+1} \sum_{l=0}^{m-1} \frac{1 - \frac{x_{i+1}^r}{(\eta^l x_j)^r}}{x_{i+1} - \eta^l x_j} + \sum_{p=1}^{m-1} k_p \sum_{l=0}^{m-1} \frac{\eta^{-pl} \eta^{-rl}}{x_{i+1}} \right) \frac{f^{s+1}}{x_1^m \cdots x_i^m x_{i+1}^r} \\
&= m(s+1 + i\kappa) \cdots m\left(s + \frac{m-r}{m} + i\kappa + k_{m-r}\right) \frac{f^{s+1}}{x_1^m \cdots x_i^m x_{i+1}^{r+1}} \\
&\dots \\
D_{i+1}^m \frac{f^{s+1}}{x_1^m \cdots x_i^m} &= m^m \left(\prod_{p=1}^m \left(s + \frac{p}{m} + i\kappa + k_p \right) \right) \frac{f^{s+1}}{x_1^m \cdots x_{i+1}^m}
\end{aligned}$$

Putting these all together yields the theorem. \square

CHAPTER 4

THE GÅRDING OPERATOR

As a student under Marcel Riesz, Gårding [Gå47] adapted the technique of analytic continuation to solve the Cauchy problem of a very special higher-order hyperbolic differential equation, and most remarkably the explicit solution manifests a stronger form of Huygens Principle: the fundamental solution is supported on a subvariety of codimension more than 1. More explicitly, the space is $\mathbb{R}^{\tilde{n}}$ of all real symmetric $n \times n$ matrices, $\tilde{n} := \frac{n(n+1)}{2}$, and the equation takes the form

$$P(\partial)u := \det \left(\varepsilon_{ij} \frac{\partial}{\partial x_{ij}} \right) u = 0$$

where $x_{ij} = x_{ji}$, $1 \leq i \leq j \leq n$, are the natural coordinates on $\mathbb{R}^{\tilde{n}}$, and

$$\varepsilon_{ij} = \begin{cases} \frac{1}{2} & i \neq j \\ 1 & i = j \end{cases}$$

The “time” direction is along the identity matrix, and it is hyperbolic in the sense that for any $\xi \in \mathbb{R}^{\tilde{n}}$, the equation $\det(\xi + tI)$ in the variable t has only real roots, a well-known fact from linear algebra. One of the crucial ingredient is again the Bernstein-Sato identity [Gå48]

$$P(\partial) \det(x_{ij})^{s+1} = (s+1) \left(s + \frac{3}{2}\right) \cdots \left(s + \frac{n+1}{2}\right) \det(x_{ij})^s$$

Even though a similar differential operator, and a similar identity, was known to Cayley as the Ω -process in classical invariant theory, I shall call it Gårding’s operator. In [Gå47], Gårding also considered a second type, where the underlying space is the space of Hermitian matrices.

It is obvious that the case of $n = 2$ is precisely the wave equation (in dimension 3 or 4), and Gårding also noted the connection with the Dirac equation of the

(massless) electron. We shall give an exposition of a few results that are not easily found in the literature. It is regrettable that the Bernstein-Sato identity does not admit a deformation by Dunkl operators, since we need $\det(x_{ij})$ to be W -invariant, but all the attempts seem to lead to a W that is no longer a reflection group.

4.1 The case of 2×2 matrices

Consider the system of partial differential equations on \mathbb{R}^3

$$\begin{pmatrix} \partial_1 & \partial_3 \\ \partial_3 & \partial_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which can be decoupled¹ by pre-multiplying the “adjoint” matrix, i.e., (the transpose of) the cofactor matrix:

$$\begin{pmatrix} \partial_2 & -\partial_3 \\ -\partial_3 & \partial_1 \end{pmatrix}$$

giving exactly the wave equation on \mathbb{R}^{1+2} for u (and same for v):

$$(\partial_1 \partial_2 - \partial_3^2) u = (\partial_t^2 - \partial_s^2 - \partial_3^2) u = 0$$

via a change of coordinates $t = x_1 + x_2$ and $s = x_1 - x_2$ (so that $\partial_1 = \partial_t + \partial_s$ and $\partial_2 = \partial_t - \partial_s$).

Naturally, the Cauchy problem for this first-order system should come with, instead of $u = u_0$ and $\partial_t u = u'_0$ on the $t = 0$ surface, the data of $u = u_0$ and $v = v_0$ on the Cauchy surface. Noting that $\partial_t u = \partial_1 u - \partial_s u = -\partial_3 v - \partial_s u$, we may write

¹This procedure is a classical one that goes as far back as Weierstrauss.

the solution u as

$$\begin{aligned}
u(t) &= \partial_t P(t) * u_0 + P(t) * u'_0 \\
&= \partial_t P(t) * u_0 + P(t) * (-\partial_s u_0 - \partial_3 v_0) \\
&= (\partial_t - \partial_s) P(t) * u_0 + (-\partial_3) P(t) * v_0 \\
&= \partial_2 P(t) * u_0 + (-\partial_3) P(t) * v_0
\end{aligned}$$

Similarly, from the fact that $\partial_t v = \partial_2 v + \partial_s v = -\partial_3 u + \partial_s v$, we have

$$\begin{aligned}
v(t) &= \partial_t P(t) * v_0 + P(t) * v'_0 \\
&= \partial_t P(t) * v_0 + P(t) * (\partial_s v_0 - \partial_3 u_0) \\
&= (\partial_t + \partial_s) P(t) * v_0 + (-\partial_3) P(t) * u_0 \\
&= \partial_1 P(t) * v_0 + (-\partial_3) P(t) * u_0
\end{aligned}$$

We may write them compactly as

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \partial_2 P(t) & -\partial_3 P(t) \\ -\partial_3 P(t) & \partial_1 P(t) \end{pmatrix} * \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

as of course it has to be! For reference,

$$P(x) = \frac{1}{2\pi\sqrt{t^2 - s^2 - x_3^2}} = \frac{1}{2\pi\sqrt{4x_1x_2 - x_3^2}} \quad \text{inside the cone } 4x_1x_2 > x_3^2$$

is not Huygens, but for $n \times n$ systems with $n \geq 3$, Gårding showed that the propagator P does satisfy the (strengthened) Huygens principle, so the system is also Huygens in the same sense.

Gårding also studied another family of hyperbolic operators, using Hermitian matrices instead of symmetric ones, and noted that the 2×2 case,

$$\begin{pmatrix} \partial_1 & \partial_3 + i\partial_4 \\ \partial_3 - i\partial_4 & \partial_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is nothing but the Dirac equation of the (massless) electron. In more familiar form,

$$\not{\partial}\psi = \begin{pmatrix} 0 & 0 & \partial_t + \partial_s & \partial_3 + i\partial_4 \\ 0 & 0 & \partial_3 - i\partial_4 & \partial_t - \partial_s \\ \partial_t - \partial_s & -\partial_3 - i\partial_4 & 0 & 0 \\ -\partial_3 + i\partial_4 & \partial_t + \partial_s & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \\ u \\ v \end{pmatrix}$$

(with Feynman's slash notation) so that $\not{\partial}\not{\partial} = \partial_t^2 - \partial_s^2 - \partial_3^2 - \partial_4^2$, the original motivation for Dirac. The propagator of this system is given by

$$\not{\partial}P = \frac{1}{4\pi} \not{\partial}\delta(t^2 - |x|^2)$$

4.2 The oscillator representation

We shall construct Gårding's operator in a round-about way. The goal is a differential operator with constant coefficient on the space $\mathbb{C}^{\tilde{n}}$ of symmetric $n \times n$ matrices, $\tilde{n} := \frac{n(n+1)}{2}$, which is invariant under the group $G' = SL(n, \mathbb{C})$ acting by $g.x = gxg^t$.

We shall introduce the oscillator representation and Howe's reductive dual pair purely on the level of Lie algebras, though some terminologies are borrowed from the Lie group actions.

Recall that for a (finite-dimensional) \mathbb{C} -vector space V endowed with a non-degenerate symmetric bilinear form $(-, -) : V \times V \rightarrow \mathbb{C}$, we may define the orthogonal Lie algebra

$$\mathfrak{o}(V) = \{X \in \text{End}_{\mathbb{C}}(V) \mid (X \cdot v, v') + (v, X \cdot v') = 0 \text{ for all } v, v' \in V\}$$

which is closed under the Lie bracket $[X, Y] = XY - YX$. On the other hand, if the non-degenerate form is anti-symmetric (for which case $\dim_{\mathbb{C}} V = 2m$ is necessarily

even), the Lie algebra so defined is called symplectic Lie algebra, denoted $\mathfrak{sp}(V)$. One may choose a polarization of the symplectic vector space, i.e. $V \cong U \oplus U^*$ such that the symplectic form is given by

$$(u + u^*, u' + u'^*) = \langle u^*, u' \rangle - \langle u'^*, u \rangle$$

for all $u, u' \in U$ and $u^*, u'^* \in U^*$. In both cases, the isomorphism type is characterized by $\dim_{\mathbb{C}} V$, hence may be denoted by \mathfrak{o}_n and \mathfrak{sp}_{2m} , respectively.

Choose a basis $\{v_i\}$ of V that respects the bilinear form, i.e., $(v_i, v_j) = \delta_{ij}$ for the orthogonal case (note that \mathbb{C} is algebraically closed), and for the symplectic case, let $\{u_i\}$ be any basis of U , and define $v_i = u_i^*$ and $v_{m+i} = u_i$, where $\{u_i^*\}$ is the basis of U^* dual to $\{u_i\}$, i.e. $\langle u_i^*, u_j \rangle = \delta_{ij}$. Then we have a standard basis E_{ij} of $\text{End}_{\mathbb{C}}(V)$, defined by $E_{ij} \cdot v_k = \delta_{jk} v_i$, that can be used to express an explicit basis of \mathfrak{o}_n and \mathfrak{sp}_{2m} . The orthogonal case is easier to describe: it is spanned by $X_{ij} = E_{ij} - E_{ji}$ for all $1 \leq i < j \leq n$.

For the symplectic case, it is more convenient to label the basis with the help of a Cartan subalgebra and the root system associated with it. Let $\mathfrak{h} \subset \mathfrak{sp}_{2m}$ be the commutative Lie subalgebra generated by

$$H_i = E_{ii} - E_{m+i, m+i} \quad i = 1, \dots, m$$

and the rest of the Lie algebra is spanned by $X_{\alpha} \in \mathfrak{sp}_{2m}$ satisfying

$$[H_i, X_{\alpha}] = \alpha(H_i) X_{\alpha} \quad \text{for all } i$$

one for each α in a finite set $\Phi \subset \mathfrak{h}^*$. The X_{α} is unique up to normalizations, but Φ can be described unambiguously. Let $\{\varepsilon_i\} \subset \mathfrak{h}^*$ be the dual basis to $\{H_i\}$, then

$$\Phi = \{\varepsilon_i + \varepsilon_j\}_{i \leq j} \cup \{-\varepsilon_i - \varepsilon_j\}_{i \leq j} \cup \{\varepsilon_i - \varepsilon_j\}_{i \neq j}$$

In particular, $\dim \mathfrak{sp}_{2m} = \dim \mathfrak{h} + |\Phi| = m(2m + 1)$.

With a choice of polarization $V = U \oplus U^*$, there is a remarkable representation of $\mathfrak{sp}(V)$ by differential operators on $U \cong \mathbb{C}^m$

$$\omega : \mathfrak{sp}(V) \rightarrow \mathcal{D}(U) \cong A_m(\mathbb{C})$$

given by

$$\begin{aligned} H_i &\mapsto \frac{1}{2}(x_i \partial_i + \partial_i x_i) = x_i \partial_i + \frac{1}{2} \\ X_{2\varepsilon_i} &\mapsto \frac{1}{2}x_i^2 & X_{\varepsilon_i + \varepsilon_j} &\mapsto x_i x_j \\ X_{-2\varepsilon_i} &\mapsto \frac{1}{2}\partial_i^2 & X_{-\varepsilon_i - \varepsilon_j} &\mapsto \partial_i \partial_j \\ X_{\varepsilon_i - \varepsilon_j} &\mapsto x_i \partial_j \end{aligned}$$

for $1 \leq i \neq j \leq m$ (for a particular normalization of X_α). We shall use the same symbol ω to denote its extension to $\mathcal{U}(\mathfrak{sp}(V))$, the universal enveloping algebra.

The fact that $[H_i, X_\alpha] = \alpha(H_i)X_\alpha$ are satisfied by the Weyl algebra elements is very natural if one recognizes that the monomials in $\mathbb{C}[x_1, \dots, x_m]$ are weight vectors for \mathfrak{h} , with weights simply the (multi-)degrees, all shifted by $\frac{1}{2}$, and each X_α acting on these monomials shifts the weights by α .

Instead of checking the other relations involving $[X_\alpha, X_\beta]$, we shall construct the “inverse map” in a coordinate-free fashion. Recall that the Weyl algebra $A = \mathcal{D}(U)$ has a filtration by total degree:

$$\mathbb{C} \cong A^0 \subset A^1 \subset \dots$$

such that $A^1/A^0 \cong U \oplus U^* \cong V$ (with its natural symplectic structure) and the associated graded algebra $\text{gr } A$ is simply the polynomial algebra $\mathbb{C}[V]$. We shall construct a map

$$A^2/A^1 \rightarrow \mathfrak{sp}(V)$$

as follows. Consider

$$A \rightarrow \text{End } A$$

$$P \mapsto \text{ad}_P = [Q \mapsto PQ - QP]$$

which restricts to a map $A^2 \rightarrow \text{End } A^1$. One should check that it descends to a map $A^2/A^1 \rightarrow \text{End } V$, and that the image is precisely $\mathfrak{sp}(V)$. It is a homomorphism of symplectic vector spaces because

$$[\text{ad}_P, \text{ad}_{P'}] = \text{ad}_{[P, P']}$$

For reference, it is straightforward to determine the normalizations of X_α so that the Weil representation ω takes the form above.

$$X_{2\varepsilon_i} = E_{m+i,i} \quad X_{\varepsilon_i+\varepsilon_j} = E_{m+i,j} + E_{m+j,i}$$

$$X_{-2\varepsilon_i} = -E_{i,m+i} \quad X_{-\varepsilon_i-\varepsilon_j} = -E_{i,m+j} - E_{j,m+i}$$

$$X_{\varepsilon_i-\varepsilon_j} = E_{ji} - E_{m+i,m+j}$$

Now, consider $V = V_1 \otimes V_2$, where V_1 is Euclidean, and V_2 is symplectic. Then, V is naturally endowed with a symplectic form given by

$$(v \otimes w, v' \otimes w') := (v, v')_{V_1} (w, w')_{V_2}$$

and a polarization $V_2 \cong U_2 \oplus U_2^*$ gives a polarization of $V \cong (V_1 \otimes U_2) \oplus (V_1 \otimes U_2)^*$. By construction, $\mathfrak{sp}(V)$ naturally contains two mutually commuting subalgebras, namely $\mathfrak{o}(V_1)$ and $\mathfrak{sp}(V_2)$, and we may write

$$\mathfrak{o}(V_1) \oplus \mathfrak{sp}(V_2) \subset \mathfrak{sp}(V)$$

the dimensions of which are

$$\frac{n(n-1)}{2} + m(2m+1) \leq nm(2nm+1)$$

respectively. Under the Weil representation

$$\omega : \mathfrak{sp}(V) \rightarrow \mathcal{D}(V_1 \otimes U_2)$$

$\mathfrak{o}(V_1)$ maps to first-order differential operators, so that

$$\mathbb{C}[V_1 \otimes U_2]^{\mathfrak{o}(V_1)} := \{f \in \mathbb{C}[V_1 \otimes U_2] \mid \omega(X) \cdot f = 0 \text{ for all } X \in \mathfrak{o}(V_1)\}$$

is a \mathbb{C} -algebra (called the algebra of invariants) on which $\mathfrak{sp}(V_2)$ acts naturally.

This yields, by restriction, a representation of $\mathfrak{sp}(V_2) \cong \mathfrak{sp}_{2m}$

$$\mathfrak{sp}(V_2) \rightarrow \mathcal{D}(\mathbb{C}[V_1 \otimes U_2]^{\mathfrak{o}(V_1)})$$

For example, if $\dim V_2 = 2$, then $V_1 \otimes U_2 \cong V_1 \cong \mathbb{C}^n$ with coordinates $x_i = v_i \otimes u^*$, and the standard basis elements $X_{ij} \in \mathfrak{o}(V_1)$ go to

$$\omega(X_{ij}) = x_i \partial_j - x_j \partial_i$$

the usual infinitesimal generators of rotations (i.e., it exponentiates to the rotation in the ij -plane of V_1), and for $\{e_+, e_-, h\}$ of $\mathfrak{sp}(V_2) \cong \mathfrak{sp}_2(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$,

$$\omega(e_+) = \frac{1}{2} \sum_{i=1}^n x_i^2 \quad \omega(e_-) = -\frac{1}{2} \sum_{i=1}^n \partial_i^2 \quad \omega(h) = \sum_{i=1}^n x_i \partial_i + \frac{n}{2}$$

are rotation-invariant differential operators. The algebra of invariants $\mathbb{C}[V_1 \otimes U_2]^{\mathfrak{o}(V_1)} \cong \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{o}_n}$ is isomorphic to the algebra $\mathbb{C}[Q]$ of an affine line, with generator $Q = x_1^2 + \dots + x_n^2$.

On the other extreme, consider $\dim V_1 = 2$, so $\mathfrak{o}(V_1) \cong \mathfrak{o}_2(\mathbb{C})$ is one-dimensional, and

$$\omega(X_{12}) = \sum_{j=1}^m x_{1j} \partial_{2j} - x_{2j} \partial_{1j}$$

where $x_{ij} = v_i \otimes u_j^*$ and $\partial_{ij} = \partial / \partial x_{ij}$. Then the algebra of invariants $\mathbb{C}[V_1 \otimes U_2]^{\mathfrak{o}(V_1)} \cong \mathbb{C}[x_{ij}]^{X_{12}}$ is generated by the “obvious” quadratic elements, namely

$$\frac{1}{2} (x_{1j}^2 + x_{2j}^2) \quad \text{and} \quad x_{1j} x_{1j'} + x_{2j} x_{2j'}$$

which shall be denoted z_{jj} and $z_{jj'}$ for $1 \leq j \neq j' \leq m$. More precisely, the algebra of invariants $\mathbb{C}[x_{ij}]^{X_{12}}$ is isomorphic to $\mathbb{C}[z_{jj}]/I_3$, where I_3 is the ideal generated by all 3×3 minors of the generic $m \times m$ symmetric matrix $\mathbf{Z} = (z_{jj'})$. In other words, it is the ring of regular functions on the subvariety $\overline{\mathcal{Z}}_2$ consisting of matrices of rank ≤ 2 .

More generally and more succinctly, for $\dim V_1 = n < m$, the algebra homomorphism

$$\begin{aligned} \varphi_n : \mathbb{C}[\mathbf{Z}] &\rightarrow \mathbb{C}[\mathbf{X}] \\ \mathbf{Z} &\mapsto \mathbf{X}^t \mathbf{X} \end{aligned}$$

factors through an isomorphism

$$\tilde{\varphi}_n : \mathbb{C}[\mathbf{Z}]/I_{n+1} \xrightarrow{\sim} \mathbb{C}[\mathbf{X}]^{\mathfrak{o}_n}$$

Here $\mathbf{X} = (x_{ij})$ and $\mathbf{Z} = ((1 + \delta_{ij})z_{ii})$ are formal matrices of order $n \times m$ and $m \times m$, respectively. This is Weyl's (First and Second) Fundamental Theorems of classical invariant theory for $O(n, \mathbb{C})$. With the identification $\tilde{\varphi}_n$, it is shown by Levasseur and Stafford that the composition

$$\omega_n : \mathcal{U}(\mathfrak{sp}_{2m}) \rightarrow \mathcal{D}(V_1 \otimes U_2)^{\mathfrak{o}(V_1)} \rightarrow \mathcal{D}(\mathbb{C}[V_1 \otimes U_2]^{\mathfrak{o}(V_1)}) \xrightarrow{\sim} \mathcal{D}(\overline{\mathcal{Z}}_n)$$

is in fact surjective for $n < m$, and $\mathcal{O}(\overline{\mathcal{Z}}_n)$ becomes a simple $\mathcal{U}(\mathfrak{sp}_{2m})$ -module. One

could express ω_n explicitly in terms of z_{ij} and $\partial_{ij} = \partial/\partial z_{ij}$

$$\omega_n(H_i) = 2z_{ii}\partial_{ii} + \sum_{j \neq i} z_{ij}\partial_{ij} + \frac{n}{2}$$

$$\omega_n(X_{2\varepsilon_i}) = z_{ii}$$

$$\omega_n(X_{\varepsilon_i + \varepsilon_j}) = z_{ij}$$

$$\omega_n(X_{\varepsilon_i - \varepsilon_j}) = z_{ij}\partial_{jj} + 2z_{ii}\partial_{ij} + \sum_{k \neq i, j} z_{ik}\partial_{jk}$$

$$\omega_n(X_{-2\varepsilon_i}) = z_{ii}\partial_{ii}^2 + \frac{n}{2}\partial_{ii} + \sum_{j \neq i} z_{ij}\partial_{ii}\partial_{ij} + \sum_{i \neq j \leq k \neq i} z_{jk}\partial_{ij}\partial_{ik}$$

$$\omega_n(X_{-\varepsilon_i - \varepsilon_j}) = z_{ij}\partial_{ij}^2 + n\partial_{ij} + \sum_{k \neq i, j} 2z_{kk}\partial_{ik}\partial_{jk} + \sum_{j \neq k \neq l \neq i} z_{kl}\partial_{ik}\partial_{jl}$$

In particular, $\mathcal{O}(\overline{\mathcal{Z}}_n) \cong \omega_n(\mathcal{U}(\mathfrak{sp}_{2m})) \cdot 1$ is a simple “lowest” weight module, with lowest weight $\frac{n}{2}(\varepsilon_1 + \cdots + \varepsilon_m)$.

Remark 4.2.1. Note the appearance of n at three places. These operators are acting on $\mathbb{C}[\mathbf{Z}]/I_{n+1}$, meaning that they preserve the ideal I_{n+1} as can be checked explicitly. Technically they only satisfy the relations of \mathfrak{sp}_{2m} when modulo I_{n+1} .

Now, if we take a different polarization of $V_1 \otimes V_2$, we may turn $\omega(\mathfrak{sp}(V_2))$ into first-order differential operators, so that the $\mathfrak{sp}(V_2)$ -invariant polynomials become an $\mathfrak{o}(V_1)$ -module. Take $\dim V_2 = 2$, and $\dim V_1 = n$ is even, and let L be an isotropic subspace of V_1 of dimension $\frac{n}{2}$. For instance, let

$$\tilde{v}_{2i-1} = \frac{v_{2i-1} + \sqrt{-1}v_{2i}}{\sqrt{2}} \quad \tilde{v}_{2i} = \frac{v_{2i-1} - \sqrt{-1}v_{2i}}{\sqrt{2}}$$

for $i = 1, \dots, \frac{n}{2}$, then $(\tilde{v}_{2i}, \tilde{v}_{2j}) = 0$ for all $1 \leq i, j \leq \frac{n}{2}$, and let L be spanned by \tilde{v}_{2i} for $i = 1, \dots, \frac{n}{2}$. Then $L \otimes V_2$ is a Lagrangian subspace of $V_1 \otimes V_2$. Then

$$\omega : \mathfrak{o}(V_1) \oplus \mathfrak{sp}(V_2) \hookrightarrow \mathfrak{sp}(V_1 \otimes V_2) \rightarrow \mathcal{D}(L \otimes V_2)$$

take the form

$$\omega(E) = \sum_{1 \leq i \leq \frac{n}{2}} x_{i1}\partial_{i2}$$

$$\begin{aligned}\omega(F) &= - \sum_{1 \leq i \leq \frac{n}{2}} x_{i2} \partial_{i1} \\ \omega(H) &= \sum_{1 \leq i \leq \frac{n}{2}} x_{i1} \partial_{i1} - x_{i2} \partial_{i2}\end{aligned}$$

Remark 4.2.2. The classical invariant theory is concerned with invariants under a group action, rather than a Lie algebra action, which results in a subtle distinction because $O(V_1)$ has two connected components, giving more symmetries than encoded in the Lie algebra $\mathfrak{o}(V_1)$. Howe observed that $O(V_1)$ and $Sp(V_2)$ not only commute, but are maximal in the sense that each is the commutant of the other inside the group $Sp(V_1 \otimes V_2)$, and Howe coined the term of (reductive) dual pair. Much of the constructions can be phrased in this setting. For instance, the map

$$\varphi : \mathbb{C}[\mathbf{Z}] \rightarrow \mathbb{C}[\mathbf{X}]$$

may be regarded as the co-morphism of a map of affine varieties

$$\begin{aligned}M_{n \times m} &\rightarrow SM_m \\ A &\mapsto A^t A\end{aligned}$$

where the image naturally lies in $\overline{\mathcal{Z}}_n \subset SM_m$, and it's also clear that it is invariant under the action of $O(n, \mathbb{C})$. This yields the easy part of the Fundamental Theorem of classical invariant theory. One may also interpret the map as

$$\begin{aligned}\mathrm{Hom}(U_2, V_1) &\rightarrow \{\text{symmetric bilinear forms } U_2 \times U_2 \rightarrow \mathbb{C}\} \\ f &\mapsto (f(-), f(-))_{V_1}\end{aligned}$$

We refer to Goodman & Wallach [GW] for the basics of classical invariant theory.

All this is in Levasseur-Stafford [LS89] in some detail, though without the explicit expressions of $\omega_n : \mathcal{U}(\mathfrak{sp}_{2m}) \rightarrow \mathcal{D}(\overline{\mathcal{Z}}_n)$.

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